SUPERSYMMETRIC GAUGE THEORIES FROM STRING THEORY

PhD-thesis of

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Wenn ich in den Grübeleien eines langen Lebens
eines gelernt habe, so ist es dies,
dass wir von einer tieferen Einsicht
in die elementaren Vorgänge viel weiter entfernt sind
als die meisten unserer Zeitgenossen glauben.

A. Einstein (1955)

Für Constanze

Abstract

The subject of this thesis are various ways to construct four-dimensional quantum field theories from string theory.

In a first part we study the generation of a supersymmetric Yang-Mills theory, coupled to an adjoint chiral superfield, from type IIB string theory on non-compact Calabi-Yau manifolds, with D-branes wrapping certain subcycles. Properties of the gauge theory are then mapped to the geometric structure of the Calabi-Yau space. In particular, the low energy effective superpotential, governing the vacuum structure of the gauge theory, can in principle be calculated from the open (topological) string theory. Unfortunately, in practice this is not feasible. Quite interestingly, however, it turns out that the low energy dynamics of the gauge theory is captured by the geometry of another non-compact Calabi-Yau manifold, which is related to the original Calabi-Yau by a geometric transition. Type IIB string theory on this second Calabi-Yau manifold, with additional background fluxes switched on, then generates a four-dimensional gauge theory, which is nothing but the low energy effective theory of the original gauge theory. As to derive the low energy effective superpotential one then only has to evaluate certain integrals on the second Calabi-Yau geometry. This can be done, at least perturbatively, and we find that the notoriously difficult task of studying the low energy dynamics of a non-Abelian gauge theory has been mapped to calculating integrals in a well-known geometry. It turns out, that these integrals are intimately related to quantities in holomorphic matrix models, and therefore the effective superpotential can be rewritten in terms of matrix model expressions. Even if the Calabi-Yau geometry is too complicated to evaluate the geometric integrals explicitly, one can then always use matrix model perturbation theory to calculate the effective superpotential.

This intriguing picture has been worked out by a number of authors over the last years. The original results of this thesis comprise the precise form of the special geometry relations on local Calabi-Yau manifolds. We analyse in detail the cut-off dependence of these geometric integrals, as well as their relation to the matrix model free energy. In particular, on local Calabi-Yau manifolds we propose a pairing between forms and cycles, which removes all divergences apart from the logarithmic one. The detailed analysis of the holomorphic matrix model leads to a clarification of several points related to its saddle point expansion. In particular, we show that requiring the planar spectral density to be real leads to a restriction of the shape of Riemann surfaces, that appears in the planar limit of the matrix model. This in turns constrains the form of the contour along which the eigenvalues have to be integrated. All these results are used to exactly calculate the planar free energy of a matrix model with cubic potential.

The second part of this work covers the generation of four-dimensional supersymmetric gauge theories, carrying several important characteristic features of the standard model, from compactifications of eleven-dimensional supergravity on G_2-manifolds. If the latter contain conical singularities, chiral fermions are present in

the four-dimensional gauge theory, which potentially lead to anomalies. We show that, locally at each singularity, these anomalies are cancelled by the non-invariance of the classical action through a mechanism called "anomaly inflow". Unfortunately, no explicit metric of a compact G_2-manifold is known. Here we construct families of metrics on compact weak G_2-manifolds, which contain two conical singularities. Weak G_2-manifolds have properties that are similar to the ones of proper G_2-manifolds, and hence the explicit examples might be useful to better understand the generic situation. Finally, we reconsider the relation between eleven-dimensional supergravity and the $E_8 \times E_8$-heterotic string. This is done by carefully studying the anomalies that appear if the supergravity theory is formulated on a ten-manifold times the interval. Again we find that the anomalies cancel locally at the boundaries of the interval through anomaly inflow, provided one suitably modifies the classical action.

Contents

Chapter 1

Setting the Stage

Wer sich nicht mehr wundern,
nicht mehr staunen kann,
der ist sozusagen tot
und sein Auge erloschen.

A. Einstein

When exactly one hundred years ago Albert Einstein published his famous articles on the theory of special relativity [50], Brownian motion [51] and the photoelectric effect [52], their tremendous impact on theoretical physics could not yet be foreseen. Indeed, the development of the two pillars of modern theoretical physics, the theory of general relativity and the quantum theory of fields, was strongly influenced by these publications. If we look back today, it is amazing to see how much we have learned during the course of the last one hundred years. Thanks to Einstein's general relativity we have a much better understanding of the concepts of space, time and the gravitational force. Quantum mechanics and quantum field theory, on the other hand, provide us with a set of physical laws which describe the dynamics of elementary particles at a subatomic scale. These theories have been tested many times, and always perfect agreement with experiment has been found [1]. Unfortunately, their unification into a quantum theory of gravity has turned out to be extremely difficult. Although it is common conviction that a unified mathematical framework describing both gravity and quantum phenomena should exist, it seems to be still out of reach. One reason for the difficulties is the fundamentally different nature of the physical concepts involved. Whereas relativity is a theory of space-time, which does not tell us much about matter, quantum field theory is formulated in a fixed background space-time and deals with the nature and interactions of elementary particles. In most of the current approaches to a theory of quantum gravity one starts from quantum field theory and then tries to extend and generalise the concepts to make them applicable to gravity. A notable exception is the field of loop quantum gravity[1], where the starting point is general

[1]For a recent review from a string theory perspective and an extensive list of references see [112].

relativity. However, given the conceptual differences it seems quite likely that one has to leave the familiar grounds of either quantum field theory or relativity and try to think of something fundamentally new.

Although many of its concepts are very similar to the ones appearing in quantum field theory, string theory [68], [117] is a branch of modern high energy physics that understands itself as being in the tradition of both quantum field theory and relativity. Also, it is quite certainly the by far most radical proposition for a unified theory. Although its basic ideas seem to be harmless - one simply assumes that elementary particles do not have a point-like but rather a string-like structure - the consequences are dramatic. Probably the most unusual prediction of string theory is the existence of ten space-time dimensions.

As to understand the tradition on which string theory is based and the intuition that is being used, it might be helpful to comment on some of the major developments in theoretical physics during the last one hundred years. Quite generally, this might be done by thinking of a physical process as being decomposable into the scene on which it takes place and the actors which participate in the play. Thinking about the scene means thinking about the fundamental nature of space and time, described by general relativity. The actors are elementary particles that interact according to a set of rules given by quantum field theory. It is the task of a unified theory to think of scene and actors as of two interdependent parts of the successful play of nature.

High energy physics in a nutshell

Special relativity tells us how we should properly think of space and time. From a modern point of view it can be understood as the insight that our world is $(\mathbb{R}^4, \eta)$, a topologically trivial four-dimensional space carrying a flat metric, i.e. one for which all components of the Riemann tensor vanish, with signature $(-,+,+,+)$. Physical laws should then be formulated as tensor equations on this four-dimensional space. In this formulation it becomes manifest that a physical process is independent of the coordinate system in which it is described.

General relativity [53] then extends these ideas to cases where one allows for a metric with non-vanishing Riemann tensor. On curved manifolds the directional derivatives of a tensor along a vector in general will not be tensors, but one has to introduce connections and covariant derivatives to be able to write down tensor equations. Interestingly, given a metric a very natural connection and covariant derivative can be constructed. Another complication that appears in general relativity is the fact that the metric itself is a dynamical field, and hence there should be a corresponding tensor field equation which describes its dynamics. This equation is know as the Einstein equation and it belongs to the most important equations of physics. It is quite interesting to note that the nature of space-time changes drastically when going from special to general relativity. Whereas in the former theory space-time is a rigid spectator on which physical theories can be formulated, this is no longer true if we allow for general metrics. The metric is both a dynamical field and it describes the space in which dynamical processes are formulated. It is this double role that makes the theory of gravity so intriguing and complicated.

Einstein's explanation of the photoelectric effect made use of the quantum nature of electro-magnetic waves, which had been the main ingredient to derive Planck's formula for black body radiation, and therefore was one major step towards the development of quantum mechanics. As is well known, this theory of atomic and subatomic phenomena was developed during the first decades of the twentieth century by Sommerfeld, Bohr, Heisenberg, Schödinger, Dirac, Pauli and many others.[2] It describes a physical system (in the Schrödinger picture) as a time-dependent state in some Hilbert space with a unitary time evolution that is determined by the Hamilton operator, which is specific to the system. Observables are represented by operators acting on the Hilbert space, and the measurable quantities are the eigenvalues of these operators. The probability (density) of measuring an eigenvalue is given by the modulus square of the system state vector projected onto the eigenspace corresponding to the eigenvalue. Shortly after the measurement the physical state is described by an eigenvector of the operator. This phenomenon is known as the collapse of the wave function and here unitarity seems to be lost. However, it is probably fair to say that the measurement process has not yet been fully understood.

In the late nineteen forties one realised that the way to combine the concepts of special relativity and quantum mechanics was in terms of a quantum theory of fields[3]. The dynamics of such a theory is encoded in an action S, and the generating functional of correlation functions is given as the path integral of $e^{\frac{i}{\hbar}S}$ integrated over all the fields appearing in the action. In the beginning these theories had been plagued by infinities, and only after these had been understood and the concept of renormalisation had been introduced did it turn into a powerful calculational tool. Scattering cross sections and decay rates of particles could then be predicted and compared to experimental data. The structure of this theoretical framework was further explored in the nineteen fifties and sixties and, together with experimental results, which had been collected in more and more powerful accelerators and detectors, culminated in the formulation of the standard model of particle physics. This theory elegantly combines the strong, the electro-magnetic and the weak force and accounts for all the particles that have been observed so far. The Higgs boson, a particle that is responsible for the mass of some of the other constituents of the model, is the only building block of the standard model that has not yet been discovered. One of the major objectives of the Large Hadron Collider (LHC), which is currently being built at CERN in Geneva, is to find it and determine its properties.

For many years all experimental results in particle physics could be explained from the standard model. However, very recently a phenomenon, known as *neutrino oscillation*, has been observed that seems to be inexplicable within this framework. The standard model contains three types of neutrinos, which do not carry charge or mass, and they only interact via the weak force. In particular, neutrinos cannot transform into each other. However, observations of neutrinos produced in the sun and the upper part of the atmosphere seem to indicate that transitions between the different types of

[2]A list of many original references can be found in [37].

[3]See [134] for a beautiful introduction and references to original work.

neutrinos do take place in nature, which is only possible if neutrinos carry mass. These experiments are not only interesting because the standard model has to be extended to account for these phenomena, but also because the mass of the neutrinos could be relevant for open questions in cosmology. Neutrinos are abundant in the universe and therefore, although their mass is tiny, they might contribute in a non-negligible way to the dark matter which is known to exist in our universe. Although interesting, since beyond the standard model, neutrino oscillation can be described by only slightly modifying the standard model action. Therefore, it does not seem to guide us in formulating a theory of quantum gravity.

There are also some theoretical facts which indicate that after all the standard model has to be modified. For some time it was believed that quantum field theories of the standard model type form the most general setting in which particle physics can be formulated. The reason is that combining some weak assumptions with the basic principles of relativity and quantum mechanics leads to no-go theorems which constrain the possible symmetries of the field theory. However, in the middle of the nineteen seventies Wess and Zumino discovered that quantum field theories might carry an additional symmetry that relates bosonic and fermionic particles and which is known as *supersymmetry* [137], [138]. It was realised very soon that the no-go theorems had been to restrictive since they required the symmetry generators to form a Lie algebra. This can be generalised to generators forming a *graded* Lie algebra and it was then shown by Haag, Lopuszanski and Sohnius that the most general graded symmetry algebra consistent with the concepts of quantum field theory is the supersymmetry algebra. This provides a strong motivation to look for supersymmetry in our world since it would in some sense be amazing if nature had chosen not to use all the freedom that it has. On the classical level supersymmetry can also be applied to theories containing the metric which are then known as supergravity theories. Of course one might ask whether it was simply the lack of this additional supersymmetry that made the problem of quantising gravity so hard. However, unfortunately it turns out that these problems persist in supergravity theories. In order to find a consistent quantum theory of gravity which contains the standard model one therefore has to proceed even further.

Another important fact, which has to be explained by a unified theory, is the difference between the Planck scale of 10^{19} GeV (or the GUT scale at 10^{16} GeV) and the preferred scale of electro-weak theory, which lies at about 10^2 GeV. The lack of understanding of this huge difference of scales is known as the *hierarchy problem*. Finally it is interesting to see what happens if one tries to estimate the value of the cosmological constant from quantum field theory. The result is by 120 orders of magnitude off the measured value! This *cosmological constant problem* is another challenge for a consistent quantum theory of gravity.

Mathematical rigour and experimental data

Physics is a science that tries to formulate abstract mathematical laws from observing natural phenomena. The experimental setup and its theoretical description are highly interdependent. However, whereas it had been the experiments that guided theoretical

insight for centuries, the situation is quite different today. Clearly, Einstein's 1905 papers were still motivated by experiments - Brownian motion and the photoelectric effect had been directly observed, and, although the Michelson-Morley experiment was much more indirect, it finally excluded the ether hypotheses thus giving way for Einstein's ground breaking theory. Similarly, quantum mechanics was developed as very many data of atomic and subatomic phenomena became available and had to be described in terms of a mathematical theory. The explanation of the energy levels in the hydrogen atom from quantum mechanics is among the most beautiful pieces of physics. The questions that are answered by quantum field theory are already more abstract. The theory is an ideal tool to calculate the results of collisions in a particle accelerator. However, modern accelerators are expensive and tremendously complicated technical devices. It takes years, sometimes decades to plan and build them. Therefore, it is very important to perform theoretical calculations beforehand and to try to predict interesting phenomena from the mathematical consistency of the theory. Based on these calculations one then has to decide which machine one should build and which phenomena one should study, in order to extract as much information on the structure of nature as possible.

An even more radical step had already been taken at the beginning of the twentieth century with the development of general relativity. There were virtually no experimental results, but relativity was developed from weak physical assumptions together with a stringent logic and an ingenious mathematical formalism. It is probably fair to say that Einstein was the first theoretical physicist in a modern sense, since his reasoning used mathematical rigour rather than experimental data.

Today physics is confronted with the strange situation that virtually any experiment can be explained from known theories, but these theories are themselves known to be incomplete. Since the gravitational force is so weak, it is very difficult to enter the regime where (classical) general relativity is expected to break down. On the other hand, the energy scales where one expects new phenomena to occur in particle physics are so high that they can only be observed in huge and expensive accelerators. The scales proposed by string theory are even way beyond energy scales that can be reached using standard machines. Therefore, today physics is forced to proceed more or less along the lines of Einstein, using pure thought and mathematical consistency. This path is undoubtedly difficult and dangerous. As we know from special and general relativity, a correct result can be extremely counterintuitive, so a seemingly unphysical theory, like string theory with its extra dimensions, should not be easily discarded. On the other hand physics has to be aware of the fact that finally its purpose is to explain experimental data and to quantitatively predict new phenomena. The importance of experiments cannot be overemphasised and much effort has to be spent in setting up ingenious experiments which might tell us something new about the structure of nature. This is a natural point where one could delve into philosophical considerations. For example one might muse about how Heisenberg's positivism has been turned on the top of its head, but I will refrain from this and rather turn to the development of string theory.

The development of string theory

String theory was originally developed as a model to understand the nature of the strong force in the late sixties. However, when Quantum Chromodynamics (QCD) came up it was quickly abandoned, with only very few people still working on strings. One of the first important developments was the insight of Joël Scherk and John Schwarz that the massless spin two particle that appears in string theory can be interpreted as the graviton, and that string theory might actually be a quantum theory of gravitation [120]. However, it was only in 1984 when string theory started to attract the attention of a wider group of theoretical physicists. At that time it had become clear that a symmetry of a classical field theory does not necessarily translate to the quantum level. If the symmetry is lost one speaks of an *anomaly*. Since local gauge and gravitational symmetries are necessary for the consistency of the theory, the requirement of anomaly freedom of a quantum field theory became a crucial issue. Building up on the seminal paper on gravitational anomalies [13] by Alvarez-Gaumé and Witten, Green and Schwarz showed in their 1984 publication [67] that $\mathcal{N}=1$ supergravity coupled to super Yang-Mills theory in ten dimensions is free of anomalies, provided the gauge group of the Yang-Mills theory is either $SO(32)$ or $E_8 \times E_8$. The anomaly freedom of the action was ensured by adding a local counterterm, now known as the *Green-Schwarz term*. Quite remarkably, both these supergravity theories can be understood as low energy effective theories of (ten-dimensional) superstring theories, namely the Type I string with gauge group $SO(32)$ and the heterotic string with gauge group $SO(32)$ or $E_8 \times E_8$. The heterotic string was constructed shortly after the appearance of the Green-Schwarz paper in [73]. This discovery triggered what is know as the first string revolution. In the years to follow string theory was analysed in great detail, and it was shown that effective theories in four dimensions with $\mathcal{N}=1$ supersymmetry can be obtained by compactifying type I or heterotic string theories on Calabi-Yau manifolds. These are compact Kähler manifolds that carry a Ricci-flat metric and therefore have $SU(3)$ as holonomy group. Four-dimensional effective actions with $\mathcal{N}=1$ supersymmetry are interesting since they provide a framework within which the above mentioned hierarchy problem can be resolved. Calabi-Yau manifolds were studied intensely, because many of their properties influence the structure of the effective four-dimensional field theory. A major discovery of mathematical interest was the fact that for a Calabi-Yau manifold X with Hodge numbers $h^{1,1}(X)$ and $h^{2,1}(X)$ there exists a *mirror manifold* Y with $h^{1,1}(Y) = h^{2,1}(X)$ and $h^{2,1}(Y) = h^{1,1}(X)$. Furthermore, it turned out that the compactification of yet another string theory, known as Type IIA, on X leads to the same effective theory in four dimensions as a fifth string theory, Type IIB, on Y. This fact is extremely useful, since quantities related to moduli of the complex structure on a Calabi-Yau manifold can be calculated from integrals in the Calabi-Yau geometry. Quantities related to the Kähler moduli on the other hand obtain corrections from world-sheet instantons and are therefore very hard to compute. Mirror symmetry then tells us that the Kähler quantities of X can be obtained from geometric integrals on Y. For a detailed exposition of mirror symmetry containing many references see [81].

Equally important for string theory was the discovery that string compactifications

on singular Calabi-Yau manifold make sense and that there are smooth paths in the space of string compactifications along which the topology of the internal manifold changes [69]. All these observations indicate that stringy geometry is quite different from the point particle geometry we are used to.

Another crucial development in string theory was Polchinski's discovery of *D-branes* [118]. These are extended objects on which open strings can end. Their existence can be inferred by exploiting a very nice symmetry in string theory, known as T-duality. In the simplest case of the bosonic string it states that string theory compactified on a circle of radius R is isomorphic to the same theory on a circle of radius $1/R$, provided the momentum quantum numbers and the winding numbers are exchanged. Note that here once again the different notion of geometry in string theory becomes apparent. One of the many reasons why D-branes are useful is that they can be used to understand black holes in string theory and to calculate their entropy.

There also has been progress in the development of quantum field theory. For example Seiberg and Witten in 1994 exactly solved the four-dimensional low-energy effective $\mathcal{N} = 2$ theory with gauge group $SU(2)$ [122]. The corresponding action is governed by a holomorphic function $\mathcal{F}$, which they calculated from some auxiliary geometry. Interestingly, one can understand this geometry as part of a Calabi-Yau compactification and the Seiberg-Witten solution can be embedded into string theory in a beautiful way.

Originally, five different consistent string theories had been constructed: Type I with gauge group $SO(32)$, Type IIA, Type IIB and the heterotic string with gauge groups $SO(32)$ and $E_8 \times E_8$. In the middle of the nineties is became clear, however, that these theories, together with eleven-dimensional supergravity, are all related by dualities and therefore are part of one more fundamental theory, that was dubbed M-theory [149]. Although the elementary degrees of freedom of this theory still remain to be understood, a lot of evidence for its existence has been accumulated. The discovery of these dualities triggered renewed interest in string theory, which is known today as the second string revolution.

Another extremely interesting duality, discovered by Maldacena and known as the AdS/CFT correspondence [101], [74], [151], relates Type IIB theory on the space $AdS_5 \times S^5$ and a four-dimensional $\mathcal{N} = 4$ supersymmetric conformal field theory on four-dimensional Minkowski space. Intuitively this duality can be understood from the fact that IIB supergravity has a brane solution which interpolates between ten-dimensional Minkowski space and $AdS_5 \times S^5$. This brane solution is thought to be the supergravity description of a D3-brane. Consider a stack of D3-branes in ten-dimensional Minkowski space. This system can be described in various ways. One can either consider the effective theory on the world-volume of the branes which is indeed an $\mathcal{N} = 4$ SCFT or one might want to know how the space backreacts on the presence of the branes. The backreaction is described by the brane solution which, close to the location of the brane, is $AdS_5 \times S^5$.

Recent developments in string theory

String theory is a vast field and very many interesting aspects have been studied in

this context. In the following a quick overview of the subjects that are going to be covered in this thesis will be given.

After it had become clear that string theories are related to eleven-dimensional supergravity, it was natural to analyse the seven-dimensional space on which one needs to compactify to obtain an interesting four-dimensional theory with the right amount of supersymmetry. It is generally expected that the four-dimensional effective field theory should live on Minkowski space and carry $\mathcal{N} = 1$ supersymmetry. Compactification to Minkowski space requires the internal manifold to carry a Ricci-flat metric. From a careful analysis of the supersymmetry transformations one finds that the vacuum is invariant under four supercharges if and only if the internal manifold carries one covariantly constant spinor. Ricci-flat seven-dimensional manifolds carrying one covariantly constant spinor are called G_2-manifolds. Indeed, one can show that their holonomy group is the exceptional group G_2. Like Calabi-Yau compactifications in the eighties, G_2-compactifications have been analysed in much detail recently. An interesting question is of course, whether one can construct standard model type theories from compactifications of eleven-dimensional supergravity on G_2-manifolds. Characteristic features of the standard model are the existence of non-Abelian gauge groups and of chiral fermions. Both these properties turn out to be difficult to obtain from G_2-compactifications. In order to generate them, one has to introduce singularities in the G_2-manifolds. Another important question, which arises once a chiral theory is constructed, is whether it is free of anomalies. Indeed, the anomaly freedom of field theories arising in string theory is a crucial issue and gives important consistency constraints. The anomalies in the context of G_2-manifolds have been analysed, and the theories have been found to be anomaly free, if one introduces an extra term into the effective action of eleven-dimensional supergravity. Interestingly, this term reduces to the standard Green-Schwarz term when compactified on a circle. Of course, this extra term is not specific to G_2-compactifications, but it has to be understood as a first quantum correction of the classical action of eleven-dimensional supergravity. In fact, it was first discovered in the context of anomaly cancellation for the M5-brane [48], [150].

Another important development that took place over the last years is the construction of realistic field theories from Type II string theory. In general, if one compactifies Type II on a Calabi-Yau manifold one obtains an $\mathcal{N} = 2$ effective field theory. However, if D-branes wrap certain cycles in the internal manifold they break half of the supersymmetry. The same is true for suitably adjusted fluxes, and so one has new possibilities to construct $\mathcal{N} = 1$ theories. Very interestingly, compactifications with fluxes and compactifications with branes turn out to be related by what is known as geometric transition. As to understand this phenomenon recall that a singularity of complex codimension three in a complex three-dimensional manifold can be smoothed out in two different ways. In mathematical language these are know as the small resolution and the deformation of the singularity. In the former case the singular point is replaced by a two-sphere of finite volume, whereas in the latter case it is replaced by a three-sphere. If the volume of either the two- or the three-sphere shrinks to zero one

obtains the singular space. The term "geometric transition" now describes the process in which one goes from one smooth space through the singularity to the other one. It is now interesting to see what happens if we compactify Type IIB string theory on two Calabi-Yau manifolds that are related by such a transition. Since one is interested in $\mathcal{N} = 1$ effective theories it is suitable to add either fluxes or branes in order to further break supersymmetry. It is then very natural to introduce D5-branes wrapping the two-spheres in the case of the small resolution of the singularity. The manifold with a deformed singularity has no suitable cycles around which D-branes might wrap, so we are forced to switch on flux in order to break supersymmetry. In fact, we can, very similar in spirit to the AdS/CFT correspondence, consider the deformed manifold as the way the geometry backreacts on the presence of the branes. This relation between string theory with flux or branes on topologically different manifolds is by itself already very exciting. However, the story gets even more interesting if we consider the effective theories generated from these compactifications. In the brane setup with N D5-branes wrapping the two-cycle we find an $\mathcal{N} = 1$ theory with gauge group $U(N)$ in four dimensions. At low energies this theory is believed to confine and the suitable description then is in terms of a chiral superfield S, which contains the gaugino bilinear. Quite interestingly, it has been shown that it is precisely this low energy effective action which is generated by the compactification on the deformed manifold. In a sense, the geometric transition and the low energy description are equivalent. In particular, the effective superpotential of the low-energy theory can be calculated from geometric integrals on the deformed manifold.

For a specific choice of manifolds the structure gets even richer. In three influential publications, Dijkgraaf and Vafa showed that IIB on the resolved manifold is related to a holomorphic matrix model. Furthermore, from the planar limit of the model one can calculate terms in the low energy effective action of the $U(N)$ $\mathcal{N} = 1$ gauge theory. More precisely, the integrals in the deformed geometry are mapped to integrals in the matrix model, where they are shown to be related to the planar free energy. Since this free energy can be calculated from matrix model Feynman diagrams, one can use the matrix model to calculate the effective superpotential.

Plan of this thesis

My dissertation is organised in two parts. In the first part I explain the intriguing connection between four-dimensional supersymmetric gauge theories, type II string theories and matrix models. As discussed above, the main idea is that gauge theories can be "geometrically engineered" from type II string theories which are formulated on the direct product of a four-dimensional Minkowski space and a six-dimensional non-compact Calabi-Yau manifold. I start by reviewing some background material on effective actions in chapter 3. Chapter 4 lists some important properties of Riemann surfaces and Calabi-Yau manifolds. Furthermore, I provide a detailed description of local Calabi-Yau manifolds, which are the spaces that appear in the context of the geometric transition. In particular, the fact that integrals of the holomorphic three-form on the local Calabi-Yau map to integrals of a meromorphic one-form on a corresponding Riemann surface is reviewed. In chapter 5 I study the holomorphic matrix model

in some detail. I show how the planar limit and the saddle point approximation have to be understood in this setup, and how special geometry relations arise. Quite interestingly, the Riemann surface that appeared when integrating the holomorphic form on a local Calabi-Yau is the same as the one appearing in the planar limit of a suitably chosen matrix model. In chapter 7 I explain why the matrix model can be used to calculate integrals on a local Calabi-Yau manifold. The reason is that there is a relation between the open B-type topological string on the Calabi-Yau and the holomorphic matrix model. All these pieces are then put together in chapter 6, where it is shown that the low energy effective action of a class of gauge theories can be obtained from integrals in the geometry of a certain non-compact Calabi-Yau manifold. After a specific choice of cycles, only one of these integral is divergent. Since the integrals appear in the formula for the effective superpotential, this divergence has to be studied in detail. In fact, the integral contains a logarithmically divergent part, together with a polynomial divergence, and I show that the latter can be removed by adding an exact term to the holomorphic three-form. The logarithmic divergence cancels against a divergence in the coupling constant, leading to a finite superpotential. Finally, I review how the matrix model can be used to calculate the effective superpotential.

In the second part I present work done during the first half of my PhD about M-theory on G_2-manifolds and (local) anomaly cancellation. I start with a short exposition of the main properties of G_2-manifolds, eleven-dimensional supergravity and anomalies. An important consistency check for chiral theories is the absence of anomalies. Since singular G_2-manifolds can be used to generate standard model like chiral theories, anomaly freedom is an important issue. In chapter 10 it is shown that M-theory on singular G_2-manifolds is indeed anomaly free. In this context it will be useful to discuss the concepts of global versus local anomaly cancellation. Then I explain the concept of weak G_2-holonomy in chapter 11, and provide examples of explicit metrics on compact singular manifolds with weak G_2 holonomy. Finally, in chapter 12 I study M-theory on $M_{10} \times I$, with I an interval, which is known to be related to the $E_8 \times E_8$ heterotic string. This setup is particularly fascinating, since new degrees of freedom living on the boundary of the space have to be introduced for the theory to be consistent. Once again a careful analysis using the concepts of local anomaly cancellation leads to new results. These considerations will be brief, since some of them have been explained rather extensively in the following review article, which is a very much extended version of my diploma thesis:

[P4] S. Metzger, *M-theory compactifications, G_2-manifolds and anomalies*, `hep-th/0308085`

In the appendices some background material is presented, which is necessary to understand the full picture. I start by explaining the notation that is being used throughout this thesis. Then I turn to some results in mathematics. The definition of divisors on Riemann surfaces is presented and the notion of relative (co-)homology is discussed. Both concepts will appear naturally in our discussion. Furthermore, I

quickly explain the Atiyah-Singer index theorem, which is important in the context of anomalies. One of the themes that seems to be omnipresent in the discussion is the concept of special geometry and of special Kähler manifolds. A detailed definition of special Kähler manifolds is given and their properties are worked out. Another central building block is the B-type topological string, and therefore I quickly review its construction. Finally, the concept of an anomaly is explained, and some of their properties are discussed.

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Schließlich danke ich Dir, Constanze.

Part I

Gauge Theories, Matrix Models and Geometric Transitions

Chapter 2

Introduction and Overview

The structure of the strong nuclear interactions is well know to be captured by Quantum Chromodynamics, a non-Abelian gauge theory with gauge group $SU(3)$, which is embedded into the more general context of the standard model of particle physics. Although the underlying theory of the strong interactions is known, it is actually very hard to perform explicit calculations in the low-energy regime of this theory. The reason is, of course, the behaviour of the effective coupling constant of QCD, which goes to zero at high energies, an effect called *asymptotic freedom*, but becomes of order 1 at energies of about 200MeV. At this energy scale the perturbative expansion in the coupling constant breaks down, and it becomes much harder to extract information about the structure of the field theory. It is believed that the theory will show a property known as *confinement*, in which the quarks form colour neutral bound states, which are the particles one observes in experiments. Most of the information we currently have about this energy regime comes from numerical calculations in lattice QCD. Pure Yang-Mills theory is also asymptotically free and it is expected to behave similarly to QCD. In particular, at low energies the massless gluons combine to colour neutral bound states, known as *glueball fields*, which are massive. Therefore, at low energies the microscopic degrees of freedom are irrelevant for a description of the theory, but it is the vacuum expectation values of composite fields which are physically interesting. These vacuum expectation values can be described by an effective potential that depends on the relevant low energy degrees of freedom. The expectation values can then be found from minimising the potential.

Understanding the low energy dynamics of QCD is a formidable problem. On the other hand, it is known that $\mathcal{N} = 1$ supersymmetric non-Abelian gauge theories share many properties with QCD. However, because of the higher symmetry, calculations simplify considerably, and some exact results can be deduced for supersymmetric theories. They might therefore be considered as a tractable toy model for QCD. In addition, we mentioned already that indications exist that the action governing physics in our four-dimensional world might actually be supersymmetric. Studying supersymmetric field theories can therefore not only teach us something about QCD but it may after all be the correct description of nature.

In order to make the basic ideas somewhat more concrete let us quickly consider

the simplest example, namely $\mathcal{N} = 1$ Yang-Mills theory with gauge group $SU(N)$. The relevant degrees of freedom at low energy are captured by a chiral superfield S which contains the gaugino bilinear. The effective superpotential, first written down by Veneziano and Yankielowicz, reads

$$W_{eff}(S) = S\left[\log\left(\frac{\Lambda^{3N}}{S^N}\right) + N\right] , \tag{2.1}$$

where Λ is the dynamical scale of the theory. The minima of the corresponding effective potential can be found by determining the critical points of the effective superpotential. Indeed, from extremising the superpotential we find

$$\langle S\rangle^N = \Lambda^{3N} , \tag{2.2}$$

which is the correct result. All this will be explained in more detail in the main text.

Over the last decades an intimate relation between supersymmetric field theories and string theory has been unveiled. Ten-dimensional supersymmetric theories appear as low energy limits of string theories and four-dimensional ones can be generated from string compactifications. The structure of a supersymmetric field theory can then often be understood from the geometric properties of the manifolds appearing in the string context. This opens up the intriguing possibility that one might actually be able to learn something about the vacuum structure of four-dimensional field theories by studying geometric properties of certain string compactifications. It is this idea that will be at the heart of the first part of this thesis.

Gauge theory - string theory duality

In fact, there is yet another intriguing relation between non-Abelian gauge theories and string theories that goes back to 't Hooft [80]. Let us consider the free energy of a non-Abelian gauge theory, which is known to be generated from connected Feynman diagrams. 't Hooft's idea was to introduce fatgraphs by representing an $U(N)$ adjoint field as the direct product of a fundamental and an anti-fundamental representation, see Fig. 2.1. The free energy can then be calculated by summing over all connected

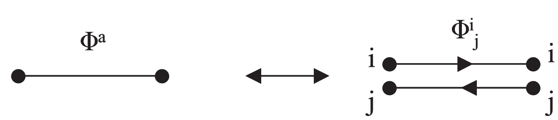

Figure 2.1: The propagator of an adjoint field Φ^a can be represented as a fatgraph. The indices i, j then run from 1 to N.

vacuum amplitudes, which are given in terms of fatgraph Feynman diagrams. By rescaling the fields one can rewrite the Lagrangian of the gauge theory in such a way that it is multiplied by an overall factor of $\frac{1}{g_{YM}^2}$. This means that every vertex comes with a factor $\frac{1}{g_{YM}^2}$, whereas the propagators are multiplied by g_{YM}^2. The gauge invariance of the Lagrangian manifests itself in the fact that all index lines form closed

(index) loops. For each index loop one has to sum over all possible indices which gives a factor of N. If E denotes the number of propagators in a given graph, V its vertices and F the number of index loops we find therefore that a given graph is multiplied by

$$\left(\frac{1}{g_{YM}^2}\right)^V \left(g_{YM}^2\right)^E N^F = (g_{YM}^2)^{E-V-F} t^F = (g_{YM}^2)^{-\chi}\, t^F = (g_{YM}^2)^{2\hat{g}-2}\, t^F\ . \tag{2.3}$$

Here we defined what is know as the *'t Hooft coupling*,

$$t := g_{YM}^2 N\ . \tag{2.4}$$

Furthermore, the Feynman diagram can be understood as the triangulation of some two-dimensional surface with F the number of faces, E the number of edges and V the number of vertices of the triangulation, see Fig. 2.2. Then the result follows

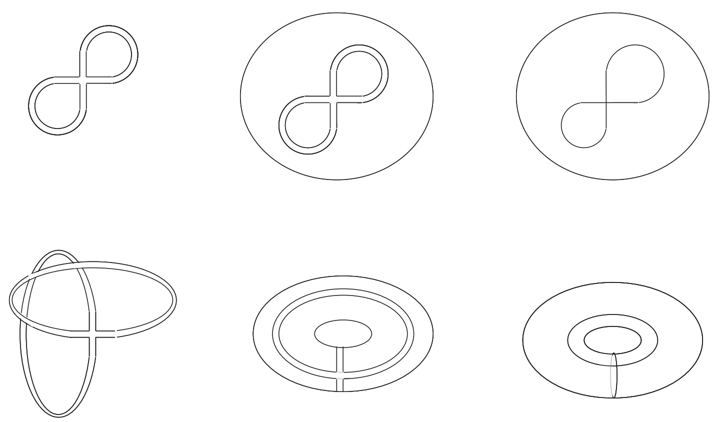

Figure 2.2: The appearance of a Riemann surface for two fatgraphs. The first graph can be drawn on a sphere and, after having shrunk the fatgraph to a standard graph, it can be understood as its triangulation with three faces, two edges and one vertex. The second graph, on the other hand, can be drawn on a torus. The corresponding triangulation has two edges, one vertex and only one face.

immediately from $V - E + F = \chi = 2 - 2\hat{g}$, where $\hat{g}$ is the genus and χ the Euler characteristic of the surface. Summing over these graphs gives the following expansion of the free energy

$$F^{gauge}\left(g_{YM}^2, t\right) = \sum_{\hat{g}=0}^{\infty} \left(g_{YM}^2\right)^{2\hat{g}-2} \sum_{h=1}^{\infty} F_{\hat{g},h} t^h + \text{non-perturbative}\ . \tag{2.5}$$

Here we slightly changed notation, using h instead of F. This is useful, since an open string theory has an expansion of precisely the same form, where now h is the number of holes in the world-sheet Riemann surface. Next we define

$$F_{\hat{g}}^{gauge}(t) := \sum_{h=1}^{\infty} F_{\hat{g},h} t^h\ , \tag{2.6}$$

which leads to

$$F^{gauge}\left(g_{YM}^2, t\right) = \sum_{\hat{g}=0}^{\infty} \left(g_{YM}^2\right)^{2\hat{g}-2} F_{\hat{g}}^{gauge}(t) + \text{non-perturbative} \ . \tag{2.7}$$

In the 't Hooft limit g_{YM}^2 is small but t is fixed, so N has to be large. The result can now be compared to the well-known expansion of the free energy of a closed string theory, namely

$$F^{string}(g_s) = \sum_{\hat{g}=0}^{\infty} g_s^{2\hat{g}-2} F_{\hat{g}}^{string} + \text{non-perturbative} \ . \tag{2.8}$$

This leads us to the obvious question whether there exists a closed string theory which, when expanded in its coupling constant g_s, calculates the free energy of our gauge theory, provided we identify

$$g_s \sim g_{YM}^2 \ . \tag{2.9}$$

In other words, is there a closed string theory such that $F_{\hat{g}}^{gauge} = F_{\hat{g}}^{string}$? Note however, that $F_{\hat{g}}^{gauge} = F_{\hat{g}}^{gauge}(t)$, so we can only find a reasonable mapping if the closed string theory depends on a parameter t.

For some simple gauge theories the corresponding closed string theory has indeed been found. The most spectacular example of this phenomenon is the AdS/CFT correspondence. Here the gauge theory is four-dimensional $\mathcal{N} = 4$ superconformal gauge theory with gauge group $U(N)$ and the corresponding string theory is Type IIB on $AdS_5 \times S^5$. However, in the following we want to concentrate on simpler examples of the gauge theory - string theory correspondence. One example is Chern-Simons theory on S^3 which is known to be dual to the A-type topological string on the resolved conifold. This duality will not be explained in detail but we will quickly review the main results at the end of this introduction. The second example is particularly simple, since the fields are independent of space and time and the gauge theory is a matrix model. To be more precise, we are interested in a holomorphic matrix model, which can be shown to be dual to the B-type closed topological string on some non-compact Calabi-Yau manifold. It will be part of our task to study these relations in more detail and to see how we can use them to extract even more information about the vacuum structure of the supersymmetric gauge theory.

Gauge theories, the geometric transition and matrix models

After these general preliminary remarks let us turn to the concrete model we want to study. Background material and many of the fine points are going to be be analysed in the main part of this thesis. Here we try to present the general picture and the relations between the various theories, see Fig. 2.3.

To be specific, we want to analyse an $\mathcal{N} = 1$ supersymmetric gauge theory with gauge group $U(N)$. Its field content is given by a vector superfield V and an adjoint

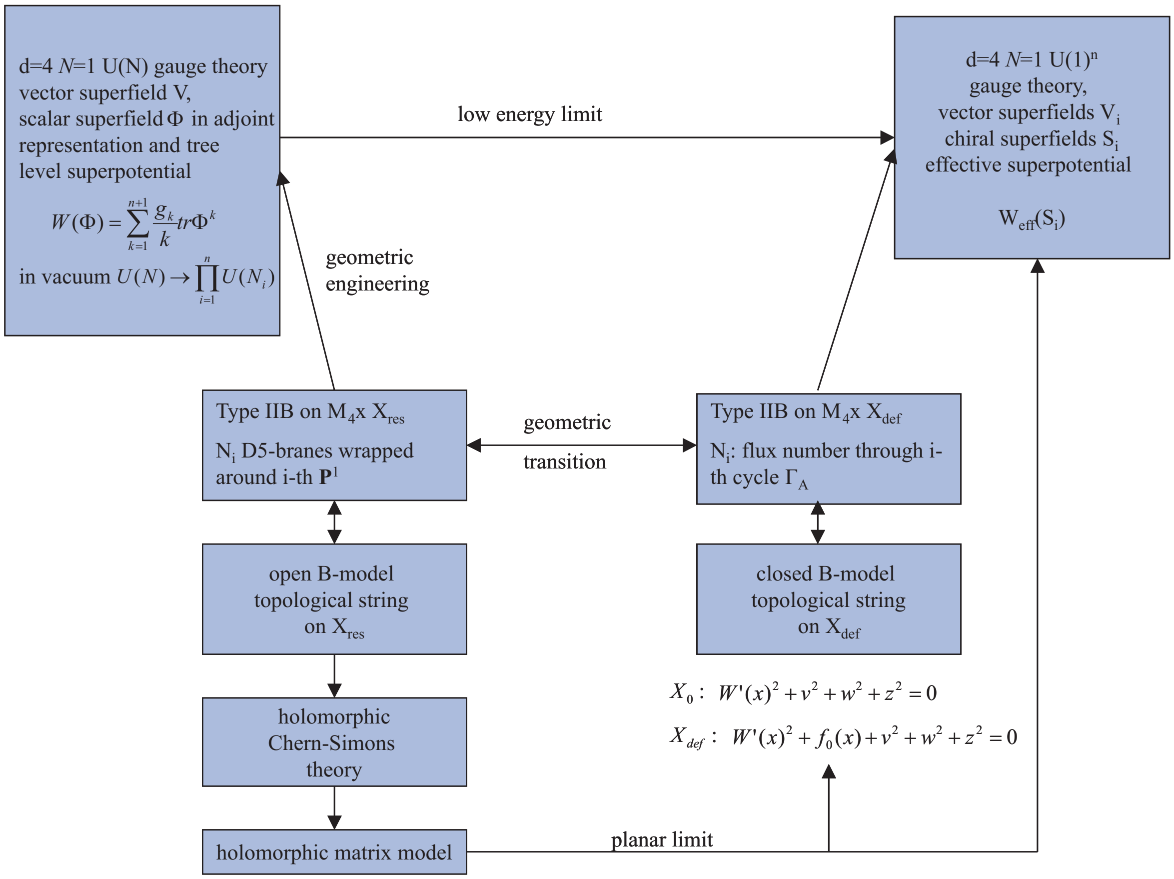

Figure 2.3: A sketch of the relation of supersymmetric Yang-Mills theory with Type IIB string theory on non-compact Calabi-Yau manifolds and with the holomorphic matrix model.

chiral superfield Φ. The dynamics of the latter is governed by the tree-level superpotential

$$W(\Phi) := \sum_{k=1}^{n+1} \frac{g_k}{k} \operatorname{tr} \Phi^k + g_0 \ , \tag{2.10}$$

with complex coefficients g_k. Here, once again, we used the equivalence of the adjoint representation of $U(N)$ and the direct product of the fundamental and anti-fundamental representation, writing $\Phi = \Phi_{ij}$. Note that the degrees of freedom are the same as those in an $\mathcal{N} = 2$ vector multiplet. In fact, we can understand the theory as an $\mathcal{N} = 2$ theory that has been broken to $\mathcal{N} = 1$ by switching on the superpotential (2.10).

It turns out that, in order to make contact with string theory, we have to expand the theory around one of its classical vacua. These vacua are obtained from distributing the eigenvalues of Φ at the critical points[1] of the superpotential, where $W'(x) = 0$.

[1] The critical points of W are always taken to be non-degenerate in this thesis, i.e. if $W'(p) = 0$ then $W''(p) \neq 0$.

Then a vacuum is specified by the numbers N_i of eigenvalues of Φ sitting at the i-th critical point of W. Note that we distribute eigenvalues at all critical points of the superpotential. Of course, we have the constraint that $\sum_{i=1}^{n} N_i = N$. In such a vacuum the gauge group is broken from $U(N)$ to $\prod_{i=1}^{n} U(N_i)$.

It is this gauge theory, in this particular vacuum, that can be generated from string theory in a process known as *geometric engineering*. To construct it one starts from Type IIB string theory on some non-compact Calabi-Yau manifold X_{res}, which is given by the small resolutions of the singular points of[2]

$$W'(x)^2 + v^2 + w^2 + z^2 = 0 \ , \tag{2.11}$$

where $W(x) = \sum_{k=1}^{n+1} \frac{g_k}{k} x^k + g_0$. This space contains precisely n two-spheres from the resolution of the n singular points. One can now generate the gauge group and break $\mathcal{N} = 2$ supersymmetry by introducing $D5$-branes wrapping these two-spheres. More precisely, we generate the theory in the specific vacuum with gauge group $\prod_i U(N_i)$, by wrapping N_i D5-branes around the i-th two-sphere. The scalar fields in Φ can then be understood as describing the position of the various branes and the superpotential is natural since D-branes have tension, i.e. they tend to wrap the minimal cycles in the non-compact Calabi-Yau manifold. The fact that, once pulled away from the minimal cycle, they want to minimise their energy by minimising their world-volume is expressed in terms of the superpotential on the gauge theory side.

Mathematically the singularity (2.11) can be smoothed out in yet another way, namely by what is know as deformation. The resulting space X_{def} can be described as an equation in $\mathbb{C}^4$,

$$W'(x)^2 + f_0(x) + v^2 + w^2 + z^2 = 0 \ , \tag{2.12}$$

where $f_0(x)$ is a polynomial of degree $n-1$. In (2.12) the n two-spheres of (2.11) have been replaced by n three-spheres. The transformation of the resolution of a singularity into its deformation is know as *geometric transition*.

Our central physical task is to learn something about the low energy limit of the four-dimensional $U(N)$ gauge theory. Since X_{res} and X_{def} are intimately related one might want to study Type IIB on X_{def}. However, the resulting effective action has $\mathcal{N} = 2$ supersymmetry and therefore cannot be related to our original $\mathcal{N} = 1$ theory. There is, however, a heuristic but beautiful argument that leads us on the right track. The geometric transition is a local phenomenon, in which one only changes the space close to the singularity. Far from the singularity an observer should not even realise that the transition takes place. We know on the other hand, that D-branes act as sources for flux and an observer far from the brane can still measure the flux generated by the brane. If the branes disappear during the geometric transition we are therefore forced to switch on background flux on X_{def}, which our observer far from the brane can measure. We are therefore led to analyse the effective field theory generated by Type IIB on X_{def} in the presence of background flux.

[2]The space X_{res} is going to be explained in detail in section 4.2, for a concise exposition of singularity theory see [15].

The four-dimensional theory is $\mathcal{N} = 1$ supersymmetric and has gauge group $U(1)^n$, i.e. it contains n Abelian vector superfields. In addition there are also n chiral superfields denoted by[3] S_i. Their scalar components describe the volumes of the n three-spheres Γ_{A_i}, which arose from deforming the singularity. Since the holomorphic $(3,0)$-form Ω, which comes with every Calabi-Yau manifold, is a calibration[4] (i.e. it reduces to the volume form on suitable submanifolds such as Γ_{A_i}) the volume can be calculated from

$$S_i = \int_{\Gamma_{A_i}} \Omega \; . \tag{2.13}$$

Type IIB string theory is known to generate a four-dimensional superpotential in the presence of three-form flux G_3, which is given by the Gukov-Vafa-Witten formula [75]

$$W_{eff}(S_i) \sim \sum_i \left(\int_{\Gamma_{A^i}} G_3 \int_{\Gamma_{B_i}} \Omega - \int_{\Gamma_{B_i}} G_3 \int_{\Gamma_{A^i}} \Omega \right) \; . \tag{2.14}$$

Here Γ_{B_i} is the three-cycle dual to Γ_{A^i}.

Now we are in the position to formulate an amazing conjecture, first written down by Cachazo, Intriligator and Vafa [27]. It simply states that *the theory generated from X_{def} in the presence of fluxes is nothing but the effective low energy description of the theory generated from X_{res} in the presence of D-branes.* Indeed, in the low energy limit one expects the $SU(N_i)$ part of the $U(N_i)$ gauge groups to confine. The theory should then be described by n chiral multiplets which contain the corresponding gaugino bilinears. The vacuum structure can be encoded in an effective superpotential. The claim is now that the n chiral superfields are nothing but the S_i and that the effective superpotential is given by (2.14). In their original publication Cachazo, Intriligator and Vafa calculated the effective superpotential directly for the $U(N)$ theory using field theory methods. On the other hand, the geometric integrals of (2.14) can be evaluated explicitly, at least for simple cases, and perfect agreement with the field theory results has been found.

These insights are very profound since a difficult problem in quantum field theory has been rephrased in a beautiful geometric way in terms of a string theory. It turns out that one can extract even more information about the field theory by making use of the relation between Type II string theory compactified on a Calabi-Yau manifold and the topological string on this Calabi-Yau. It has been known for a long time that the topological string calculates terms in the effective action of the Calabi-Yau compactification [14], [20]. For example, if we consider the Type IIB string we know that the vector multiplet part of the four-dimensional effective action is determined from the prepotential of the moduli space of complex structures of the Calabi-Yau

[3] Later on we will introduce fields $S_i,\ \bar{S}_i,\ \tilde{S}_i$ with slightly different definitions. Since we are only interested in a sketch of the main arguments, we do not distinguish between these fields right now. Also, S_i sometimes denotes the full chiral multiplet, and sometimes only its scalar component. It should always be clear from the context, which of the two is meant.

[4] A precise definition of calibrations and calibrated submanifolds can be found in [85].

manifold. But this function is nothing else than the genus zero free energy of the corresponding B-type topological string. Calculating the topological string free energy therefore gives information about the effective field theory. In the case we are interested in, with Type IIB compactified on X_{res} with additional D5-branes, one has to study the open B-type topological string with topological branes wrapping the two-cycles of X_{res}. It can be shown [148] that in this case the corresponding string field theory reduces to holomorphic Chern-Simons theory and, for the particular case of X_{res}, this was shown by Dijkgraaf and Vafa [43] to simplify to a holomorphic matrix model with partition function

$$Z = C_{\hat{N}} \int \mathrm{d}M \exp\left(-\frac{1}{g_s} \operatorname{tr} W(M)\right) , \tag{2.15}$$

where the potential $W(x)$ is given by the same function as the superpotential above. Here g_s is a coupling constant, $\hat{N}$ is the size of the matrices and $C_{\hat{N}}$ is some normalisation constant. Clearly, this is a particularly simple and tractable theory and one might ask whether one can use it to calculate interesting physical quantities. The holomorphic matrix model had not been studied until very recently [95], and in our work [P5] some more of its subtleties have been unveiled. Similarly to the case of a Hermitean matrix model one can study the planar limit in which the size $\hat{N}$ of the matrices goes to infinity, the coupling g_s goes to zero and the product $t := g_s\hat{N}$ is taken to be fixed. In this limit there appears a Riemann surface of the form

$$y^2 = W'(x)^2 + f_0(x) , \tag{2.16}$$

which clearly is intimately related to X_{def}. Indeed, as we will see below, the integrals in the geometry of X_{def}, which appear in (2.14), can be mapped to integrals on the Riemann surface (2.16). These integrals in turn can be related to the free energy of the matrix model at genus zero, $\mathcal{F}_0$. After this series of steps one is left with an explicit formula for the effective superpotential,

$$W_{eff}(S) \sim \sum_{i=1}^{n} \left(N_i \frac{\partial \mathcal{F}_0(S)}{\partial S_i} - S_i \log \Lambda_i^{2N_i} - 2\pi i S_i \tau \right) , \tag{2.17}$$

where dependence on S means dependence on all the S_i. The constants Λ_i and τ will be explained below. The free energy can be decomposed into a perturbative and a non-perturbative part (c.f. Eq. (2.7)). Using monodromy arguments one can show that $\frac{\partial \mathcal{F}_0^{np}}{\partial S_i} \sim S_i \log S_i$, and therefore

$$W_{eff}(S) \sim \sum_{i=1}^{n} \left(N_i \frac{\partial \mathcal{F}_0^p(S)}{\partial S_i} + N_i S_i \log\left(\frac{S_i}{\Lambda_i^2}\right) - 2\pi i S_i \tau \right) , \tag{2.18}$$

where now $\mathcal{F}_0^p$ is the perturbative part of the free energy at genus zero, i.e. we can calculate it by summing over all the planar matrix model vacuum amplitudes. This gives a perturbative expansion of W_{eff}, which upon extremisation gives the vacuum gluino condensate $\langle S \rangle$. Thus using a long chain of dualities in type IIB string theory

we arrive at the beautiful result that the low energy dynamics and vacuum structure of a non-Abelian gauge theory can be obtained from perturbative calculations in a matrix model.[5]

Chern-Simons theory and the Gopakumar-Vafa transition
In the following chapters many of the points sketched so far are going to be made more precise by looking at the technical details and the precise calculations. However, before starting this endeavour, it might be useful to give a quick overview of what happens in the case of IIA string theory instead. As a matter of fact, in this context very many highly interesting results have been uncovered over the last years. The detailed exposition of these developments would certainly take us too far afield, but we consider it nevertheless useful to provide a quick overview of the most important results. For an excellent review including many references to original work see [103]. In fact, the results in the context of IIB string theory on which we want to report have been discovered only after ground breaking work in IIA string theory. The general picture is quite similar to what happens in Type IIB string theory, and it is sketched in Fig. 2.4.

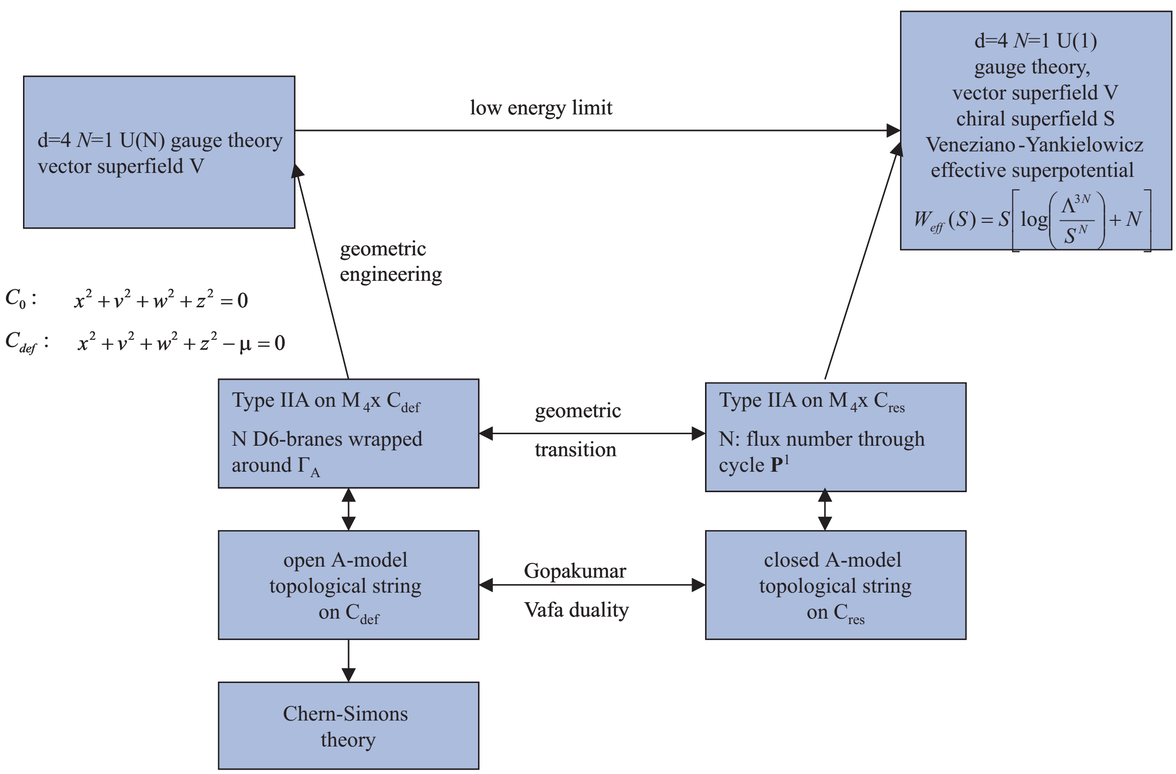

Figure 2.4: A sketch of the relation of supersymmetric Yang-Mills theory with Type IIA string theory on the conifold and with Chern-Simons theory.

[5]In fact, this result can also be proven without making use of string theory and the geometric transition, see [42], [26].

The starting point here is to consider the open A-model topological string on T^*S^3 with topological branes wrapped around the three-cycle at the center. Witten's string field theory then does not reduce to holomorphic Chern-Simons theory, as is the case on the B-side, but to ordinary Chern-Simons theory [148], [145] on S^3,

$$S = \frac{k}{4\pi} \int_{S^3} \left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A \right) . \tag{2.19}$$

To be more precise, Witten showed that the $F^{CS}_{\hat{g},h}$ of the expansion (2.5) for Chern-Simons theory on S^3 equals the free energy of the open A-model topological string on T^*S^3 at genus $\hat{g}$ and h holes, $F^{A-tst}_{\hat{g},h}$. The details of this procedure can be found in [148].

Of course, the space T^*S^3 is isomorphic to the deformed conifold, a space we want to call C_{def}, which is given by

$$x^2 + v^2 + w^2 + z^2 = \mu \ . \tag{2.20}$$

Clearly, this is the deformation of the singularity

$$x^2 + v^2 + w^2 + z^2 = 0 \ . \tag{2.21}$$

As we have seen above, these singularities can be smoothed out in yet another way, namely by means of a small resolution. The resulting space is known as the resolved conifold and will be denoted by C_{res}. Both C_{def} and C_{res} are going to be studied in detail in 4.2. Motivated by the AdS/CFT correspondence, in which a stack of branes in one space has a dual description in some other space without branes (but with fluxes), in [64], [65], [66] Gopakumar and Vafa studied whether there exists a dual closed string description of the open topological string on C_{def}, and hence of Chern-Simons theory. This turns out to be the case and the dual theory is given by the closed topological A-model on C_{res}. To be somewhat more precise, the Gopakumar-Vafa conjecture states that Chern-Simons gauge theory on S^3 with gauge group $SU(N)$ and level k is equivalent to the closed topological string of type A on the resolved conifold, provided we identify

$$g_s = \frac{2\pi}{k+N} = g^2_{CS} \quad , \quad \kappa = \frac{2\pi i N}{k+N} \ , \tag{2.22}$$

where κ is the Kähler modulus of the two-sphere appearing in the small resolution. Note that $\kappa = it$ where $t = g^2_{CS} N$ is the 't Hooft coupling.

On the level of the partition function this conjecture was tested in [66]. Here we only sketch the main arguments, following [103]. The partition function $Z^{CS}(S^3) = \exp\left(-F^{CS}\right)$ of $SU(N)$ Chern-Simons theory on S^3 is known, including non-perturbative terms [145]. The free energy splits into a perturbative $F^{CS,p}$ and a non-perturbative piece $F^{CS,np}$, where the latter can be shown to be

$$F^{CS,np} = \log \frac{(2\pi g_s)^{\frac{1}{2}N^2}}{\mathrm{vol}\,(U(N))} \ . \tag{2.23}$$

We find that the non-perturbative part of the free energy comes from the volume of the gauge group in the measure. As discussed above, the perturbative part has an expansion

$$F^{CS,p} = \sum_{\hat{g}=0}^{\infty} g_s^{2\hat{g}-2} \sum_{h=1}^{\infty} F_{\hat{g},h} t^h \ . \tag{2.24}$$

The sum over h can actually be performed and gives $F_{\hat{g}}^{CS,p}$. The non-perturbative part can also be expanded in the string coupling and, for $g \geq 2$, the sum of both pieces leads to (see e.g. [103])

$$F_{\hat{g}}^{CS} = F_{\hat{g}}^{CS,p} + F_{\hat{g}}^{CS,np} = \frac{(-1)^{\hat{g}}|B_{2\hat{g}}B_{2\hat{g}-2}|}{2\hat{g}(2\hat{g}-2)(2\hat{g}-2)!} + \frac{|B_{2\hat{g}}|}{2\hat{g}(2\hat{g}-2)!}\mathrm{Li}_{3-2\hat{g}}\left(e^{-\kappa}\right) \ , \tag{2.25}$$

where B_n are the Bernoulli numbers and $\mathrm{Li}_j(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^j}$ is the polylogarithm of index j.

This result can now be compared to the free energy of the topological A-model on the resolved conifold. Quite generally, from results of [32], [19], [20], [64], [65] and [54] it can be shown that the genus $\hat{g}$ contribution to the free energy of the topological A-model on a Calabi-Yau manifold X reads [103]

$$\begin{aligned} F_{\hat{g}}^{A-tst} &= \frac{(-1)^{\hat{g}}\chi(X)|B_{2\hat{g}}B_{2\hat{g}-2}|}{4\hat{g}(2\hat{g}-2)(2\hat{g}-2)!} \\ &\quad + \sum_{\beta}\left(\frac{|B_{2\hat{g}}|n_\beta^0}{2\hat{g}(2\hat{g}-2)!} + \frac{2(-1)^{\hat{g}}n_\beta^2}{(2\hat{g}-2)!} \pm \ldots - \frac{\hat{g}-2}{12}n_\beta^{\hat{g}-1} + n_\beta^{\hat{g}}\right)\mathrm{Li}_{3-2\hat{g}}\left(Q^\beta\right) \ . \end{aligned} \tag{2.26}$$

Here $\beta = \sum_i n_i[\gamma_i^{(2)}]$ is a homology class, where the $[\gamma_i^{(2)}]$ form a basis of $H_2(X)$. In general there are more than one Kähler parameters κ_i and Q^β has to be understood as $\prod_i Q_i^{n_i}$ with $Q_i := e^{-\kappa_i}$. Furthermore, the $n_\beta^{\hat{g}}$, known as *Gopakumar-Vafa invariants*, are integer numbers.

We see that precise agreement can be found between (2.25) and (2.26), provided we set $\chi(X) = 2$ and $n_1^0 = 1$, with all other Gopakumar-Vafa invariants vanishing. This is indeed the correct set of geometric data for the resolved conifold, and we therefore have shown that the conjecture holds, at least at the level of the partition function. In order to have a full duality between two theories, however, one should not only compare the free energy but also the observables, which in Chern- Simons theory are given by Wilson loops. In [113] the corresponding quantities were constructed in the A-model string, thus providing further evidence for the conjecture. Finally, a nice and intuitive proof of the duality from a world-sheet perspective has been given by the same authors in [114]. Quite interestingly this duality can be lifted to non-compact G_2-manifolds, where the transition is a flop [4], [16].

After having established the Gopakumar-Vafa duality we can now proceed similarly to the above discussion on the IIB side. Indeed, in [130] the duality was embedded into the context of full string theory. There the statement is that IIA string theory on the

direct product of four-dimensional Minkowski space and the deformed conifold with N D6-branes wrapping around the S^3 in T^*S^3 is dual to the IIA string on Minkowski times the resolved conifold, where one now has to switch on flux with flux number N through the S^2. As above one can also study the four-dimensional effective field theories generated by these compactifications. The N D6-branes clearly lead to pure supersymmetric Yang-Mills theory with gauge group $U(N)$, whereas the dual theory on C_{res} with fluxes switched on, leads to an effective $U(1)$ theory in four dimensions. In [130] it is also shown that the effective superpotential generated from IIA on the resolved conifold is nothing but the Veneziano-Yankielowicz potential. Thus, like in the IIB case sketched above, the geometric transition is once again equivalent to the low energy description.

Both the resolved and the deformed conifold can be described in the language of toric varieties. In fact, one can study the A-type topological string on more general toric varieties. The geometry of these spaces can be encoded in terms of toric diagrams and the geometric transition then has a nice diagrammatic representation. Quite interestingly, it was shown in [10] that, at least in principle, one can compute the partition function of the A-type topological string on any toric variety. This is done by understanding the toric diagram as some sort of "Feynman diagram", in the sense that to every building block of the diagram one assigns a mathematical object and the partition function corresponding to a toric variety is then computed by putting these mathematical objects together, following a simple and clear cut set of rules. Many more results have been derived in the context of the A-type topological string on toric varieties, including the relation to integrable models [9]. These developments are, however, outside the scope of my thesis.

Chapter 3

Effective Actions

In what follows many of the details of the intriguing picture sketched in the introduction will be explained. Since the full picture consists of very many related but different theories we will not be able to study all of them in full detail. However, we are going to provide references wherever a precise explanation will not be possible. Here we start by an exposition of various notions of effective actions that exist in quantum field theory. We quickly review the definitions of the generating functional of the one-particle-irreducible correlation functions and explain how it can be used to study vacua of field theories. The Wilsonian effective action is defined somewhat differently, and it turns out that it is particularly useful in the context of supersymmetric gauge theories. Finally a third type of effective action is presented. It is defined in such a way that it captures the symmetries and the vacuum structure of the theory, and in some but not in all cases it coincides with the Wilsonian action.

3.1 The 1PI effective action and the background field method

We start from the Lagrangian density of a field theory

$$\mathcal{L} = \mathcal{L}(\Phi) \tag{3.1}$$

and couple the fields $\Phi(x)$ to a set of classical currents $J(x)$,

$$Z[J] = \exp(iF[J]) = \int D\Phi \ \exp\left(i\int \mathrm{d}^4x \ \mathcal{L}(\Phi) + i\int \mathrm{d}^4x \ J(x)\Phi(x)\right) \ . \tag{3.2}$$

The quantity $iF[J]$ is the sum of all connected vacuum-vacuum amplitudes. Define

$$\Phi_J(x) := \frac{\delta}{\delta J(x)} F[J] = \langle \Phi(x) \rangle_J \ . \tag{3.3}$$

This equation can also be used to define a current $J_{\Phi_0}(x)$ for a given classical field $\Phi_0(x)$, s.t. $\Phi_J(x) = \Phi_0(x)$ if $J(x) = J_{\Phi_0}(x)$. Then one defines the *quantum effective*

action $\Gamma[\Phi_0]$ as

$$\Gamma[\Phi_0] := F[J_{\Phi_0}] - \int \mathrm{d}^4x\ \Phi_0(x) J_{\Phi_0}(x)\ . \tag{3.4}$$

The functional $\Gamma[\Phi_0]$ is an effective action in the sense that $iF[J]$ can be calculated as a sum of connected *tree* graphs for the vacuum-vacuum amplitude, with vertices calculated as if $\Gamma[\Phi_0]$ and not $S[\Phi]$ was the action. But this implies immediately that $i\Gamma[\Phi_0]$ is the sum of all one-particle-irreducible (1PI) connected graphs with arbitrary number of external lines, each external line corresponding to a factor of Φ_0. Another way to put it is (see for example [134] or [116] for the details)

$$\exp(i\Gamma[\Phi_0]) = \int_{1PI} D\phi\ \exp(iS[\Phi_0 + \phi])\ . \tag{3.5}$$

Furthermore, varying Γ gives

$$\frac{\delta\Gamma[\Phi_0]}{\delta\Phi_0(x)} = -J_{\Phi_0}(x)\ , \tag{3.6}$$

and in the absence of external currents

$$\frac{\delta\Gamma[\Phi_0]}{\delta\Phi_0(x)} = 0\ . \tag{3.7}$$

This can be regarded as the equation of motion for the field Φ_0, where quantum corrections have been taken into account. In other words, it determines the stationary configurations of the background field Φ_0.

The *effective potential* of a quantum field theory is defined as the non-derivative terms of its effective Lagrangian. We are only interested in translation invariant vacua, for which $\Phi_0(x) = \Phi_0$ is constant and one has

$$\Gamma[\Phi_0] = -V_4\ V_{eff}(\Phi_0)\ , \tag{3.8}$$

where V_4 is the four-dimensional volume of the space-time in which the theory is formulated.

To one loop order the 1PI generating function can be calculated from

$$\exp(i\Gamma[\Phi_0]) \approx \int D\phi\ \exp\left(i\int \mathrm{d}^4x\ \mathcal{L}^q(\Phi_0,\phi)\right)\ , \tag{3.9}$$

$\mathcal{L}^q(\Phi_0,\phi)$ contains all those terms of $\mathcal{L}(\Phi_0+\phi)$ that are at most quadratic in ϕ. One writes $\mathcal{L}^q(\Phi_0,\phi) = \mathcal{L}(\Phi_0) + \tilde{\mathcal{L}}^q(\Phi_0,\phi)$ and performs the Gaussian integral over ϕ, which schematically leads to

$$\Gamma[\Phi_0] \approx V_4\mathcal{L}(\Phi_0) - \frac{1}{2}\log\det(A(\Phi_0))\ , \tag{3.10}$$

where A is the matrix of second functional derivatives of $\mathcal{L}$ with respect to the fields ϕ.[1] For some concrete examples the logarithm of this determinant can be calculated

[1] Since we have to include only 1PI diagrams to calculate Γ (c.f. Eq. 3.5) the terms linear in the fluctuations ϕ do not contribute to Γ. See [2] for a nice discussion.

and one can read off the effective action and the effective potential to one loop order. Minimising this potential then gives the stable values for the background fields, at least to one loop order. The crucial question is, of course, whether the structure of the potential persists if higher loop corrections are included.

Example: Yang-Mills theory
In order to make contact with some of the points mentioned in the introduction, we consider the case of Yang-Mills theory with gauge group $SU(N)$ and Lagrangian

$$\mathcal{L} = -\frac{1}{4g^2} F^a_{\mu\nu} F^{\mu\nu a} \ , \tag{3.11}$$

and try to determine its effective action using the procedure described above. Since the details of the calculation are quite complicated we only list the most important results. One start by substituting

$$A^a_\mu(x) \to A^a_\mu(x) + a^a_\mu(x) \tag{3.12}$$

into the action and chooses a gauge fixing condition. This condition is imposed by adding gauge fixing and ghost terms to the action. Then, the effective action can be evaluated to one loop order from

$$\exp(i\Gamma[A]) \approx \int Da\ Dc\ D\bar{c}\ \exp\left(i \int \mathrm{d}^4x\ \mathcal{L}^q(A, a, c, \bar{c})\right) \ , \tag{3.13}$$

where $\mathcal{L}^q(A, a, c, \bar{c})$ only contains those terms of $\mathcal{L}(A + a) + \mathcal{L}^{gf} + \mathcal{L}^{ghost}$ that are at most quadratic in the fields $a, c, \bar{c}$. Here $\mathcal{L}^{gf}$ is the gauge fixing and $\mathcal{L}^{ghost}$ the ghost Lagrangian. The Gaussian integrals can then be evaluated and, at least for small N, one can work out the structure of the determinant (see for example chapter 17.5. of [134]). The form of the potential is similar in shape to the famous Mexican hat potential, which implies that the perturbative vacuum where one considers fluctuations around the zero-field background is an unstable field configuration. The Yang-Mills vacuum lowers its energy by spontaneously generating a non-zero ground state.

However, this one-loop calculation can only be trusted as long as the effective coupling constant is small. On the other hand, from the explicit form of the effective action one can also derive the one-loop β-function. It reads

$$\beta(g) = -\frac{\frac{11}{3} N g^3}{16\pi^2} + \ldots \ , \tag{3.14}$$

where the dots stand for higher loop contributions. The renormalisation group equation

$$\mu \frac{\partial}{\partial \mu} g(\mu) = \beta(g) \tag{3.15}$$

is solved by

$$\frac{1}{g(\mu)^2} = -\frac{\frac{11}{3} N}{8\pi^2} \log\left(\frac{|\Lambda|}{\mu}\right) \ . \tag{3.16}$$

Therefore, for energies lower or of order $|\Lambda|$ we cannot trust the one-loop approximation. Nevertheless, computer calculations in lattice gauge theories seem to indicate that even for small energies the qualitative picture remains true, and the vacuum of Yang-Mills theory is associated to a non-trivial background field configuration, which gives rise to confinement and massive glueball fields. However, the low energy physics of non-Abelian gauge theories is a regime which has not yet been understood.

3.2 Wilsonian effective actions of supersymmetric theories

For a given Lagrangian one can also introduce what is known as the *Wilsonian effective action* [141], [142]. Take λ to be some energy scale and define the Wilsonian effective Lagrangian $\mathcal{L}_\lambda$ as the local Lagrangian that, with λ imposed as an ultraviolet cut-off, reproduces precisely the same results for S-matrix elements of processes at momenta below λ as the original Lagrangian $\mathcal{L}$. In general, masses and coupling constants in the Wilsonian action will depend on λ and usually there are infinitely many terms in the Lagrangian. Therefore, the Wilsonian action might not seem very attractive. However, it can be shown that its form is quite simple in the case of supersymmetric theories.

Supersymmetric field theories are amazingly rich and beautiful. Independently on whether they turn out to be the correct description of nature, they certainly are useful to understand the structure of quantum field theory. This is the case since they often possess many properties and characteristic features of non-supersymmetric field theories, but the calculations are much more tractable, because of the higher symmetry. For an introduction to supersymmetry and some background material see [21], [134], [135]. Here we explain how one can calculate the Wilsonian effective superpotential in the case of $\mathcal{N}=1$ supersymmetric theories.

The $\mathcal{N}=1$ supersymmetric action of a vector superfield V coupled to a chiral superfield Φ transforming under some representation of the gauge group is given by[2]

$$S=\int \mathrm{d}^4x\,[\Phi^\dagger e^{-V}\Phi]_D-\int \mathrm{d}^4x\,\left[\left(\frac{\tau}{16\pi i}\operatorname{tr} W^\tau\epsilon W\right)_F+c.c.\right]+\int \mathrm{d}^4x\,\left[(W(\Phi))_F+c.c.\right]\,, \tag{3.17}$$

where $W(\Phi)$ is known as the (tree-level) *superpotential*. The subscripts F and D extract the F- respectively D-component of the superfield in the bracket. Renormalisability forces W to be at most cubic in Φ, but since we are often interested in theories which can be understood as effective theories of some string theory, the condition of renormalisability will often be relaxed.[3] The constant τ is given in terms of the bare

[2]We follow the notation of [134], in particular $[\operatorname{tr} W^\tau\epsilon W]_F=[\epsilon_{\alpha\beta}\operatorname{tr} W_\alpha W_\beta]_F=\frac{1}{2}\operatorname{tr} F_{\mu\nu}F^{\mu\nu}-\frac{i}{4}\epsilon_{\mu\nu\rho\sigma}\operatorname{tr} F^{\mu\nu}F^{\rho\sigma}+\operatorname{tr}\bar\lambda\partial\!\!\!/(1-\gamma_5)\lambda-\operatorname{tr} D^2$. Here $F_{\mu\nu}$ etc. are to be understood as $F^a_{\mu\nu}t^a$, where t^a are the Hermitean generators of the gauge group, which satisfy $\operatorname{tr} t^at^b=\delta^{ab}$.

[3]Of course, for non-renormalisable theories the first two terms can have a more general structure as well. The first term, for instance, in general reads $K(\Phi,\Phi^\dagger e^{-V})$ where K is known as the Kähler potential. However, these terms presently are not very important for us. See for example [135], [134] for the details.

coupling g and the Θ-angle,

$$\tau = \frac{4\pi i}{g^2} + \frac{\Theta}{2\pi} \ . \tag{3.18}$$

The ordinary bosonic potential of the theory reads

$$V(\phi) = \sum_n \left|\frac{\partial W}{\partial \phi_n}\right|^2 + \frac{g^2}{2}\sum_a \left(\sum_{mn} \phi_n^* \phi_m (t^a)_{mn}\right)^2 \ , \tag{3.19}$$

where ϕ is the lowest component of the superfield Φ and t^a are the Hermitean generators of the gauge group. A supersymmetric vacuum ϕ_0 of the theory is a field configuration for which V vanishes [134], so we have the so called F-flatness condition

$$\frac{\partial}{\partial \phi} W(\phi)\bigg|_{\phi_0} = 0 \ , \tag{3.20}$$

as well as the D-flatness condition

$$\sum_{mn} \phi_n^* \phi_m (t^a)_{mn}\bigg|_{\phi_0} = 0 \ . \tag{3.21}$$

The space of solutions to the D-flatness condition is known as the *classical moduli space* and it can be shown that is can always be parameterised in terms of a set of independent holomorphic gauge invariants $X_k(\phi)$.

The task is now to determine the effective potential of this theory in order to learn something about its quantum vacuum structure. Clearly, one possibility is to calculate the 1PI effective action, however, for supersymmetric gauge theories there exist non-renormalisation theorems which state that the Wilsonian effective actions of these theories is particularly simple.

Proposition 3.1 *Perturbative non-renormalisation theorem*
If the cut-off λ appearing in the Wilsonian effective action preserves supersymmetry and gauge invariance, then the Wilsonian effective action to all orders in perturbation theory has the form

$$S_\lambda = \int \mathrm{d}^4x \ [(W(\Phi))_F + c.c.] - \int \mathrm{d}^4x \ \left[\left(\frac{\tau_\lambda}{16\pi i}\,\mathrm{tr}\, W^\tau \epsilon W\right)_F + c.c.\right] + D\text{-}terms \ , \tag{3.22}$$

where

$$\tau_\lambda = \frac{4\pi i}{g_\lambda^2} + \frac{\Theta}{2\pi} \ , \tag{3.23}$$

and g_λ is the one-loop effective coupling.

Note in particular that the superpotential remains unchanged in perturbation theory, and that the gauge kinetic term is renormalised only at one loop. The theorem was proved in [72] using supergraph techniques, and in [121] using symmetry arguments and analyticity.

Although the superpotential is not renormalised to any finite order in perturbation theory it does in fact get corrected on the non-perturbative level, i.e. one has

$$W_{eff} = W_{tree} + W_{non-pert} \ . \tag{3.24}$$

The non-perturbative contributions were thoroughly studied in a series of papers by Affleck, Davis, Dine and Seiberg [41], [8], using dimensional analysis and symmetry considerations. For some theories these arguments suffice to exactly determine $W_{non-pert}$. An excellent review can be found in [134]. This effective Wilsonian superpotential can now be used to study the quantum vacua of the gauge theory, which have to be critical points of the effective superpotential.

So far we defined two effective actions, the generating functional of 1PI amplitudes and the Wilsonian action. Clearly, it is important to understand the relation between the two. In fact, for the supersymmetric theories studied above one can also evaluate the 1PI effective action. It turns out that this functional receives contributions to all loop orders in perturbation theory, corresponding to Feynman diagrams in the background fields with arbitrarily many internal loops. Therefore, we find that the difference between the two effective actions seems to be quite dramatic. One of them is corrected only at one-loop and the other one obtains corrections to all loop orders. The crucial point is that one integrates over all momenta down to zero to obtain the 1PI effective action, but one only integrates down to the scale λ to calculate the Wilsonian action. In other words, whereas one has to use tree-diagrams only if one is working with the 1PI effective action, one has to include loops in the Feynman diagrams if one uses the Wilsonian action. However, the momentum in these loops has an ultraviolet cutoff λ. Taking this λ down to zero then gives back the 1PI generating functional. Therefore, the difference between the two has to come from the momentum domain between 0 and λ. Indeed, as was shown by Shifman and Vainshtein in [123], in supersymmetric theories the two-loop and higher contributions to the 1PI effective action are infrared effects. They only enter the Wilsonian effective action as the scale λ is taken to zero. For finite λ the terms in the Wilsonian effective action arise only from the tree-level and one-loop contributions, together with non-perturbative corrections.

Furthermore, it turns out that the fields and coupling constants that appear in the Wilsonian effective action are *not* the physical quantities one would measure in experiment. For example, the non-renormalisation theorem states that the coupling constant g is renormalised only at one loop. However, from explicit calculations one finds that the 1PI g is renormalised at all loops. This immediately implies that there are two different coupling constants, the Wilsonian one and the 1PI coupling. The two are related in a non-holomorphic way and again the difference can be shown to come from infrared effects. It is an important fact that the Wilsonian effective superpotential does depend holomorphically on both the fields and the (Wilsonian) coupling constants, whereas the 1PI effective action is non-holomorphic in the (1PI) coupling constants. The relation between the two quantities has been pointed out in [124], [46]. In fact, one can be brought into the other by a non-holomorphic change of variables. Therefore, for supersymmetric theories we can confidently use the Wilsonian effective superpotential

to study the theory. If the non-perturbative corrections to the superpotential are calculable (which can often be done using symmetries and holomorphy) then one can obtain the exact effective superpotential and therefore exact results about the vacuum structure of the theory. However, the price one has to pay is that this beautiful description is in terms of unphysical Wilsonian variables. The implications for the true physical quantities can only be found after undoing the complicated change of variables.

3.3 Symmetries and effective potentials

There is yet another way (see [84] for a review and references), to calculate an effective superpotential, which uses Seiberg's idea [121] to interpret the coupling constants as chiral superfields. Let

$$W(\Phi) = \sum_k g_k X_k(\Phi) \tag{3.25}$$

be the tree-level superpotential, where the X_k are gauge invariant polynomials in the matter chiral superfield Φ. In other words, the X_k are themselves chiral superfields. One can now regard the coupling constants g_k as the vacuum expectation value of the lowest component of another chiral superfield G_k, and interpret this field as a source [121]. I.e. instead of (3.25) we add the term $W(G,\Phi) = \sum_k G_k X_k$ to the action. Integrating over Φ then gives the partition function $Z[G] = \exp(iF[G])$. If we assume that supersymmetry is unbroken F has to be a supersymmetric action of the chiral superfields G, and therefore it can be written as

$$F[G] = \int \mathrm{d}^4x \ \ [(W_{low}(G))_F + c.c.] + \dots \ , \tag{3.26}$$

with some function $W_{low}(G)$. As we will see, this function can often be determined from symmetry arguments. For standard fields (i.e. not superfields) we have the relation (3.3). In a supersymmetric theory this reads

$$\langle X_k \rangle_G = \frac{\delta}{\delta G_k} F[G] = \frac{\partial}{\partial G_k} W_{low}(G) \tag{3.27}$$

where we used that W_{low} is holomorphic in the fields G_k. On the other hand we can use this equation to define $\langle G_k \rangle$ as the solution of

$$X_k^0 = \frac{\partial}{\partial G_k} W_{low}(G) \bigg|_{\langle G_k \rangle} \ . \tag{3.28}$$

Then we define the Legendre transform of W_{low}

$$W_{dyn}(X_k^0) := W_{low}(\langle G_k \rangle) - \sum_k \langle G_k \rangle X_k^0 \ , \tag{3.29}$$

where $\langle G_k \rangle$ solves (3.28), and finally we set

$$W_{eff}(X_k, g_k) := W_{dyn}(X_k) + \sum_k g_k X_k \ . \tag{3.30}$$

This effective potential has the important property that the equations of motion for the fields X_k derived from it determine their expectation values. Note that (3.30) is nothing but the tree-level superpotential corrected by the term W_{dyn}. This looks similar to the Wilsonian superpotential, of which we know that it is uncorrected perturbatively but it obtains non-perturbative corrections. Indeed, for some cases the Wilsonian superpotential coincides with (3.30), however, in general this is not the case (see [84] for a discussion of these issues). Furthermore, since W_{dyn} does not depend on the couplings g_k, the effective potential depends linearly on g_k. This is sometimes known as the *linearity principle*, and it has some interesting consequences. For instance one might want to integrate out the field X_i by solving

$$\frac{\partial W_{eff}}{\partial X_i} = 0 \ , \tag{3.31}$$

which can be rewritten as

$$g_i = -\frac{\partial W_{dyn}}{\partial X_i} \ . \tag{3.32}$$

If one solves this equation for X_i in terms of g_i and the other variables and plugs the result back in W_{eff}, the g_i-dependence will be complicated. In particular, integrating out all the X_i gives back the superpotential $W_{low}(g)$. However, during this process one does actually not loose any information, since this procedure of integrating out X_i can actually be inverted by *integrating in* X_i. This is obvious from the fact that, because of the linearity in g_k, integrating out X_i is nothing but performing an (invertible) Legendre transformation.

Super Yang-Mills theory and the Veneziano-Yankielowicz potential

In order to see how the above recipe is applied in practice, we study the example of $\mathcal{N} = 1$ Super-Yang-Mills theory. Its action reads

$$S_{SYM} = -\int \mathrm{d}^4x \ \left[\left(\frac{\tau}{16\pi i} \operatorname{tr} \ W^\tau \epsilon W \right)_F + c.c. \right] \ , \tag{3.33}$$

which, if one defines the chiral superfield

$$S := \frac{1}{32\pi^2} \operatorname{tr} \ W^\tau \epsilon W \ , \tag{3.34}$$

can be rewritten as

$$S_{SYM} = \int \mathrm{d}^4x \ \left[(2\pi i \tau S)_F + c.c. \right] \ . \tag{3.35}$$

S is known as the *gaugino bilinear superfield*, whose lowest component is proportional to $\lambda\lambda \equiv \operatorname{tr} \lambda^\tau \epsilon \lambda$. Note that both S and τ are complex.

The classical action (3.33) is invariant under a chiral $U(1)$ R-symmetry that acts as $W_\alpha(x,\theta) \to e^{i\varphi} W_\alpha(x, e^{-i\varphi}\theta)$, which implies in particular that $\lambda \to e^{i\varphi}\lambda$. The quantum theory, however, is not invariant under this symmetry, which can be understood from the fact that the measure of the path integral is not invariant. The phenomenon in which a symmetry of the classical action does not persist at the quantum level is known as an anomaly. For a detailed analysis of anomalies and some applications in string and M-theory see the review article [P4]. The most important results on anomalies are listed in appendix E. The precise transformation of the measure for a general transformation $\lambda \to e^{i\epsilon(x)}\lambda$ can be evaluated [134] and reads

$$D\lambda D\bar{\lambda} \to D\lambda' D\bar{\lambda}' = \exp\left(i \int \mathrm{d}^4x \; \epsilon(x) G[x;A] \right) D\lambda D\bar{\lambda} \tag{3.36}$$

where

$$G[x;A] = -\frac{N}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} F_a^{\mu\nu} F_a^{\rho\sigma} \; . \tag{3.37}$$

For the global R-symmetry $\epsilon(x) = \varphi$ is constant, and $\frac{1}{64\pi^2}\int \mathrm{d}^4x \; \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} = \nu$ is an integer, and we see that the symmetry is broken by instantons. Note that, because of the anomaly, the chiral rotation $\lambda \to e^{i\varphi}\lambda$ is equivalent to $\Theta \to \Theta - 2N\varphi$ (c.f. Eq. (3.18)). Thus, the chiral rotation is a symmetry only if $\varphi = \frac{k\pi}{N}$ with $k = 0, \ldots 2N-1$, and the $U(1)$ symmetry is broken to $\mathbb{Z}_{2N}$.

The objective is to study the vacua of $\mathcal{N}=1$ super Yang-Mills theory by probing for gaugino condensates, to which we associate the composite field S that includes the gaugino bilinear. This means we are interested in the effective superpotential $W_{eff}(S)$, which describes the symmetries and anomalies of the theory. In particular, upon extremising $W_{eff}(S)$ the value of the gaugino condensate in a vacuum of $\mathcal{N}=1$ Yang-Mills is determined.

The β-function of $\mathcal{N}=1$ super Yang-Mills theory reads at one loop

$$\beta(g) = -\frac{3Ng^3}{16\pi^2} \tag{3.38}$$

and the solution of the renormalisation group equation is given by

$$\frac{1}{g^2(\mu)} = -\frac{3N}{8\pi^2} \log \frac{|\Lambda|}{\mu} \; . \tag{3.39}$$

Then

$$\tau_{1-loop} \;=\; \frac{4\pi i}{g^2(\mu)} + \frac{\Theta}{2\pi} = \frac{1}{2\pi i} \log \left(\frac{|\Lambda| e^{i\Theta/3N}}{\mu} \right)^{3N} =: \frac{1}{2\pi i} \log \left(\frac{\Lambda}{\mu} \right)^{3N} \tag{3.40}$$

enters the one-loop action

$$S_{1-loop} = \int \mathrm{d}^4x \;\; [(2\pi i \tau_{1-loop} S)_F + c.c.] = \int \mathrm{d}^4x \;\; \left[\left(3N \log \left(\frac{\Lambda}{\mu} \right) S \right)_F + c.c. \right] \; . \tag{3.41}$$

In order to determine the superpotential $W_{low}(\tau)$ one can use Seiberg's method [121], and one interprets τ as a background chiral superfield. This is useful, since the effective superpotential is known to depend *holomorphically* on all the fields, and therefore it has to depend holomorphically on τ. Furthermore, once τ is interpreted as a field, spurious symmetries occur. In the given case one has the spurious R-symmetry transformation

$$\begin{aligned} W(x,\theta) &\rightarrow e^{i\varphi}W(x,e^{-i\varphi}\theta) \ , \\ \tau &\rightarrow \tau + \frac{N\varphi}{\pi} \ , \end{aligned} \tag{3.42}$$

and the low energy potential has to respect this symmetry. This requirement, together with dimensional analysis, constrains the superpotential W_{low} uniquely to

$$W_{low} = N\mu^3 \exp\left(\frac{2\pi i}{N}\tau\right) = N\Lambda^3 \ , \tag{3.43}$$

where μ has dimension one. Indeed, this $W_{low}(\tau)$ transforms as $W_{low}(\tau) \rightarrow e^{2i\varphi}W_{low}(\tau)$, and therefore the action is invariant.

Before deriving the effective action let us first show that a non-vanishing gaugino condensate exists. One starts from (3.33) and now one treats τ as a background field. Then, since $W_{\alpha a} = \lambda_{\alpha a} + \ldots$, the F-component of τ, denoted by τ_F, acts as a source for $\lambda\lambda$. One has

$$\begin{aligned} \langle\lambda\lambda\rangle &= \frac{1}{Z}\int D\Phi\ e^{iS}\lambda\lambda = \frac{1}{Z}\int D\Phi\ \exp\left[-i\int d^4x\left[\left(\frac{\tau}{16\pi i}W^\tau\epsilon W\right)_F + c.c.\right]\right]\lambda\lambda \\ &= -16\pi\frac{1}{Z}\int D\Phi\ \frac{\delta}{\delta\tau_F}e^{iS} = -\frac{16\pi}{Z}\frac{\delta}{\delta\tau_F}Z = -16\pi\frac{\delta}{\delta\tau_F}\log Z \\ &= -16\pi i\frac{\delta}{\delta\tau_F}\int \mathrm{d}^4x\ \left[(W_{low})_F + c.c.\right] + \ldots = -16\pi i\frac{\partial}{\partial\tau}W_{low}(\tau) \\ &= 32\pi^2\mu^3\exp\left(\frac{2\pi i\tau}{N}\right) \ , \end{aligned} \tag{3.44}$$

where $D\Phi$ stands for the path integral over all the fields. τ is renormalised only at one loop and non-perturbatively, and it has the general form

$$\tau = \frac{3N}{2\pi i}\log\left(\frac{\Lambda}{\mu}\right) + \sum_{n=1}^{\infty} a_n\left(\frac{\Lambda}{\mu}\right)^{3Nn} \ . \tag{3.45}$$

Therefore, the non-perturbative terms of τ only contribute to a phase of the gaugino condensate, and it is sufficient to plug in the one-loop expression for τ. The result is a non-vanishing gaugino condensate,

$$\langle\lambda\lambda\rangle = 32\pi^2\Lambda^3 \ . \tag{3.46}$$

The presence of this condensate means that the vacuum does not satisfy the $\mathbb{Z}_{2N}$ symmetry, since $\langle\lambda\lambda\rangle \rightarrow e^{2i\varphi}\langle\lambda\lambda\rangle$ and only a $\mathbb{Z}_2$-symmetry survives. The remaining

$|\frac{\mathbb{Z}_{2N}}{\mathbb{Z}_2}| - 1 = N - 1$ transformations, i.e. those with $\varphi = \frac{k\pi}{N}$ with $k = 1, \ldots N-1$ transform one vacuum into another one, and we conclude that there are N distinct vacua.

Next we turn to the computation of the effective action. From (3.35) we infer that $2\pi i S$ and τ are conjugate variables. We apply the recipe of the last section, starting from $W_{low}(\tau)$, as given in (3.43). From

$$2\pi i S = \frac{\partial}{\partial \tau} W_{low}(\tau)\bigg|_{\langle\tau\rangle} = 2\pi i \mu^3 \exp\left(\frac{2\pi i}{N}\langle\tau\rangle\right) \tag{3.47}$$

we infer that $\langle\tau\rangle = \frac{N}{2\pi i}\log\left(\frac{S}{\mu^3}\right)$. According to (3.29) one finds

$$W_{dyn}(S) = NS - NS\log\left(\frac{S}{\mu^3}\right) \ . \tag{3.48}$$

Finally, using the one-loop expression for τ and identifying the μ appearing in (3.43) with the one in (3.40), one obtains the Veneziano-Yankielowicz potential [132]

$$W_{eff}(\Lambda, S) = W_{VY}(\Lambda, S) = S\left[N + \log\left(\frac{\Lambda^{3N}}{S^N}\right)\right] \ . \tag{3.49}$$

In order to see in what sense this is the correct effective potential, one can check whether it gives the correct expectation values. Indeed, $\frac{\partial W_{eff}(\Lambda,S)}{\partial S}\Big|_{\langle S\rangle} = 0$ gives the N vacua of $\mathcal{N}=1$ super Yang-Mills theory,

$$\langle S\rangle = \Lambda^3 e^{\frac{2\pi i k}{N}} \ , \quad k = 0, \ldots N-1 \ . \tag{3.50}$$

Note that this agrees with (3.46), since S is defined as $\frac{1}{32\pi^2}$ times $\operatorname{tr} W_a W^\alpha$, which accounts for the difference in the prefactors. Furthermore, the Veneziano-Yankielowicz potential correctly captures the symmetries of the theory. Clearly, under R-symmetry S transforms as $S \to \tilde{S} = e^{2i\varphi}S$ and therefore the effective Veneziano-Yankielowicz action $\mathcal{S}_{VY}$ transforms as

$$\begin{aligned}
\mathcal{S}_{VY} \to \tilde{\mathcal{S}}_{VY} &= \int \mathrm{d}^4x \ \left[\left(\tilde{W}_{VY}\right)_{\tilde{F}} + c.c.\right] + \ldots \\
&= \int \mathrm{d}^4x \ \left[e^{2i\varphi}\left(SN + S\log\left(\frac{\Lambda^{3N}}{S^N}\right) - 2iN\varphi S\right)_{\tilde{F}} + c.c.\right] \\
&= \mathcal{S}_{VY} - \int \mathrm{d}^4x \ \left[\left(\frac{iN\varphi}{16\pi^2}\operatorname{tr} W^\tau \epsilon W\right)_F + c.c.\right] \ ,
\end{aligned} \tag{3.51}$$

which reproduces the anomaly. However, for the effective theory to have the $\mathbb{Z}_{2N}$ symmetry, one has to take into account that the logarithm is a multi-valued function. For the n-th branch one must define

$$W_{VY}^{(n)}(\Lambda, S) = S\left[N + \log\left(\frac{\Lambda^{3N}}{S^N}\right) + 2\pi i n\right] \ . \tag{3.52}$$

Then the discrete symmetry shifts $\mathcal{L}_n \to \mathcal{L}_{n-k}$ for $\varphi = \frac{\pi k}{N}$. The theory is invariant under $\mathbb{Z}_{2N}$ if we define

$$Z = \sum_{n=-\infty}^{\infty} \int DS \exp\left(i \int \mathrm{d}^4x \, \left[W_{VY}^{(n)} + c.c. \right]_F + \ldots \right) . \tag{3.53}$$

Thus, although the Veneziano-Yankielowicz effective action is not the Wilsonian effective action, it contains all symmetries, anomalies and the vacuum structure of the theory.

Super Yang-Mills coupled to matter

The main objective of the next chapters is to find an effective superpotential in the Veneziano-Yankielowicz sense, i.e. one that is not necessarily related to the 1PI or Wilsonian effective action, but that can be used to find the vacuum structure of the theory, for $\mathcal{N} = 1$ Yang-Mills theory coupled to a chiral superfield Φ in the adjoint representation. The tree-level superpotential in this case is given by

$$W(\Phi) = \sum_{k=1}^{n+1} \frac{g_k}{k} \operatorname{tr} \Phi^k + g_0 \ , \ g_k \in \mathbb{C} \tag{3.54}$$

where without loss of generality $g_{n+1} = 1$. Furthermore, the critical points of W are taken to be non-degenerate, i.e. if $W'(p) = 0$ then $W''(p) \neq 0$. As we mentioned above, the corresponding effective superpotential can be evaluated perturbatively from a holomorphic matrix model [43], [44], [45]. This can either be shown using string theory arguments based on the results of [130] and [27], or from an analysis in field theory [42], [26]. The field theory itself has been studied in [28], [29], [57]. However, before we turn to explaining these development we need to present background material on the manifolds appearing in this context.

Chapter 4

Riemann Surfaces and Calabi-Yau Manifolds

In this section we explain some elementary properties of Riemann surfaces and Calabi-Yau manifolds. Of course, both types of manifolds are ubiquitous in string theory and studying them is of general interest. Here we will concentrate on those aspects that are relevant for our setup. As we mentioned in the introduction, the theory we are interested in can be geometrically engineered by "compactifying" Type II string theory on non-compact Calabi-Yau manifolds, the structure of which will be presented in detail. The superpotential can be calculated from geometric integrals of a three-form over a basis of three-cycles in these manifolds. Quite interestingly, it turns out that the non-compact Calabi-Yau manifolds are intimately related to Riemann surfaces and that the integrals on the Calabi-Yau can be mapped to integrals on the Riemann surface. As we will see in the next chapter, it is precisely this surface which also appears in the large $\hat{N}$ limit of a holomorphic matrix model.

Our main reference for Riemann surfaces is [55]. An excellent review of both Riemann surfaces and Calabi-Yau manifolds, as well as their physical applications can be found in [81]. The moduli space of Calabi-Yau manifolds was first studied in [34].

4.1 Properties of Riemann surfaces

Definition 4.1 A *Riemann surface* is a complex one-dimensional connected analytic manifold.

There are many different description of Riemann surfaces. We are interested in the so called hyperelliptic Riemann surfaces of genus $\hat{g}$,

$$y^2 = \prod_{i=1}^{\hat{g}+1}(x - a_i^+)(x - a_i^-) \ , \tag{4.1}$$

with $x, y, a_i^\pm \in \mathbb{C}$ and all $a_i^\pm$ different. These can be understood as two complex sheets glued together along cuts running between the branch points a_i^- and a_i^+, together

with the two points at infinity of the two sheets, denoted by Q on the upper and Q' on the lower sheet. On these surfaces the set $\left\{\frac{\mathrm{d}x}{y}, \frac{x\mathrm{d}x}{y}, \ldots, \frac{x^{\hat{g}-1}\mathrm{d}x}{y}\right\}$ forms a basis of holomorphic differentials. This fact can be understood by looking at the theory of divisors on Riemann surfaces, presented in appendix B.2, see also [55]. The divisors capture the zeros and divergences of functions on the Riemann surface. Let $P_1, \ldots P_{2\hat{g}+2}$ denote those points on the Riemann surface which correspond to the zeros of y (i.e. to the $a_i^\pm$). Close to $a_i^\pm$ the good coordinates are $z_i^\pm = \sqrt{x - a_i^\pm}$. Then the divisor of y is given by $\frac{P_1 \ldots P_{2\hat{g}+2}}{Q^{\hat{g}+1} Q'^{\hat{g}+1}}$, since y has simple zeros at the P_i in the good coordinates z_i, and poles of order $\hat{g}+1$ at the points Q, Q'. If we let R, R' denote those points on the Riemann surface which correspond to zero on the upper, respectively lower sheet then it is clear that the divisor of x is given by $\frac{RR'}{QQ'}$. Finally, close to $a_i^\pm$ we have $\mathrm{d}x \sim z_i^\pm \mathrm{d}z_i^\pm$, and obviously $\mathrm{d}x$ has double poles at Q and Q', which leads to a divisor $\frac{P_1 \ldots P_{2\hat{g}+2}}{Q^2 Q'^2}$. In order to determine the zeros and poles of more complicated objects like $\frac{x^k \mathrm{d}x}{y}$ we can now simply multiply the divisors of the individual components of this object. In particular, the divisor of $\frac{\mathrm{d}x}{y}$ is $Q^{\hat{g}-1} Q'^{\hat{g}-1}$, and for $\hat{g} \geq 1$ it has no poles. Similarly, we find that $\frac{x^k \mathrm{d}x}{y}$ has no poles if $k \leq \hat{g}-1$. Quite generally, for any compact Riemann surface Σ of genus $\hat{g}$ one has,

$$\dim \mathsf{Hol}^1_{\hat{g}}(\Sigma) = \hat{g} \ , \tag{4.2}$$

where $\mathsf{Hol}^1_{\hat{g}}(\Sigma)$ is the first holomorphic de Rham cohomology group on Σ with genus $\hat{g}$. On the other hand, later on we will be interested in integrals of the form $y\mathrm{d}x$, with divisor $\frac{P_1^2 \ldots P_{2\hat{g}+2}^2}{Q^{\hat{g}+3} Q'^{\hat{g}+3}}$, showing that $y\mathrm{d}x$ has poles of order $\hat{g}+3, \hat{g}+2, \ldots 1$ at Q and Q'. To allow for such forms with poles one has to mark points on the surface. This marking amounts to pinching a hole into the surface. Then one can allow forms to diverge at this point, since it no longer is part of the surface. The dimension of the first holomorphic de Rham cohomology group on Σ of genus $\hat{g}$ with n marked points, $\mathsf{Hol}^1_{\hat{g},n}(\Sigma)$, is given by [55]

$$\dim \mathsf{Hol}^1_{\hat{g},n}(\Sigma) = 2\hat{g} + n - 1 \ . \tag{4.3}$$

The first homology group $H_1(\Sigma; \mathbb{Z})$ of the Riemann surface Σ has $2\hat{g}$ generators $\alpha^i, \beta_i,\ i \in \{1, \ldots, \hat{g}\}$, with intersection matrix

$$\begin{aligned} &\alpha^i \cap \alpha^j = \emptyset \quad , \quad \beta_i \cap \beta_j = \emptyset \ , \\ &\alpha^i \cap \beta_j = -\beta_j \cap \alpha^i = \delta^i_j \ . \end{aligned} \tag{4.4}$$

Note that this basis is defined only up to a[1] $Sp(2\hat{g}, \mathbb{Z})$ transformation. To see this formally, consider the vector $v := (\alpha^i\ \beta_j)^\tau$ that satisfies $v \cap v^\tau = \mho$. But for $S \in Sp(2\hat{g}, \mathbb{Z})$ we have for $v' := Sv$ that $v' \cap v'^\tau = S \mho S^\tau = \mho$. A possible choice of the cycles α^i, β_i for the hyperelliptic Riemann surface (4.1) is given in Fig. 4.1. As we

[1]There are different conventions for the definition of the symplectic group in the literature. We

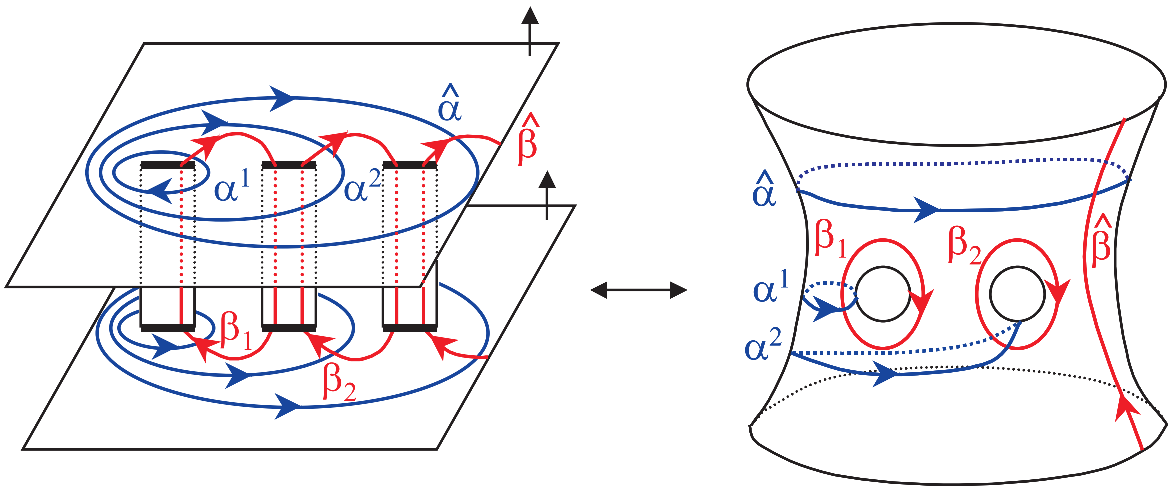

Figure 4.1: The hyperelliptic Riemann surface (4.1) can be understood as two complex sheets glued together along cuts running between a_i^- and a_i^+. Here we indicate a symplectic set of cycles for $\hat{g}=2$. It consists of $\hat{g}$ compact cycles α^i, surrounding i of the cuts, and their compact duals β_i, running from cut i to $i+1$ on the upper sheet and from cut $i+1$ to cut i on the lower one. We also indicated the relative cycles $\hat{\alpha},\hat{\beta}$, which together with α^i,β_i form a basis of the relative homology group $H_1(\Sigma,\{Q,Q'\})$. Note that the orientation of the two planes on the left-hand side is chosen such that both normal vectors point to the top. This is why the orientation of the α-cycles is different on the two planes. To go from the representation of the Riemann surface on the left to the one on the right one has to flip the upper plane.

mentioned already, later on we will be interested in integrals of $y\mathrm{d}x$, which diverges at Q,Q'. Therefore, we are led to consider a Riemann surface with these two points excised. On such a surface there exists a very natural homology group, namely the relative homology $H_1(\Sigma,\{Q,Q'\})$. For a detailed exposition of relative (co-)homology see appendix B.3. $H_1(\Sigma,\{Q,Q'\})$ not only contains the closed cycles α^i,β_i, but also a cycle $\hat{\beta}$, stretching from Q to Q', together with its dual $\hat{\alpha}$. As an example one might look at the simple Riemann surface

$$y^2 = x^2 - \mu = (x-\sqrt{\mu})(x-\sqrt{\mu}) \ , \tag{4.7}$$

with only one cut between $-\sqrt{\mu}$ and $\sqrt{\mu}$ surrounded by a cycle $\hat{\alpha}$. The dual cycle $\hat{\beta}$ simply runs from Q' through the cut to Q.

There are various symplectic bases of $H_1(\Sigma,\{Q,Q'\})$. Next to the one just presented, another set of cycles often appears in the literature. It contains $\hat{g}+1$ compact

adopt the following:

$$Sp(2m,\mathbb{K}) := \{S\in GL(2m,\mathbb{K}) : S^\tau \mho S = \mho\}, \tag{4.5}$$

where

$$\mho := \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \tag{4.6}$$

and $\mathbb{1}$ is the $m\times m$ unit matrix. $\mathbb{K}$ stands for any field.

cycles A^i, each surrounding one cut only, and their duals B_i, which are all non-compact, see Fig. 4.2. Although string theory considerations often lead to this basis, it turns

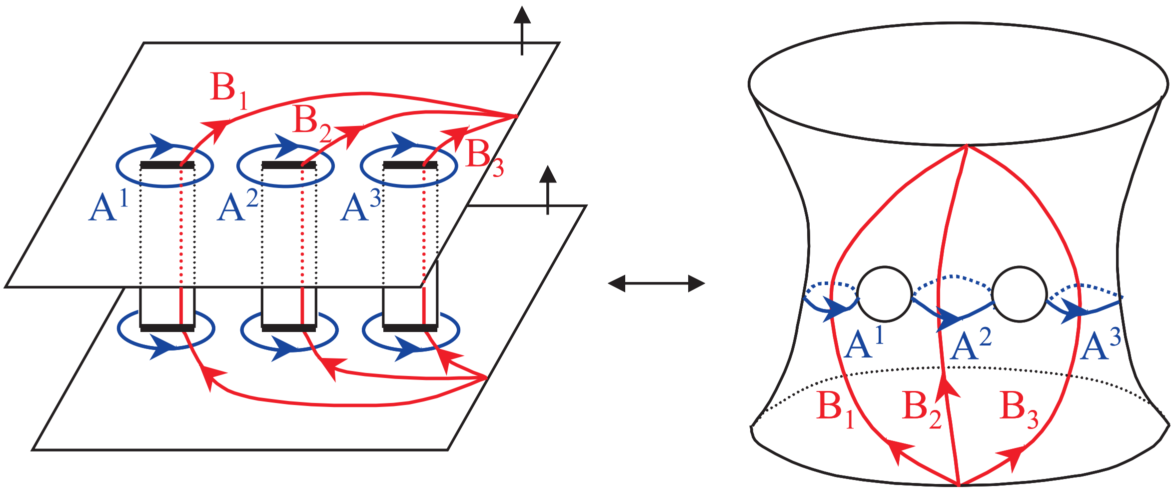

Figure 4.2: Another choice of basis for $H_1(\Sigma, \{Q, Q'\})$ containing compact A-cycles and non-compact B-cycles.

out to be less convenient, basically because of the non-compactness of the B-cycles.

Next we collect a couple of properties which hold for any (compact) Riemann surface Σ. Let ω be any one-form on Σ, then

$$\begin{pmatrix} \int_{\alpha^i} \omega \\ \int_{\beta_j} \omega \end{pmatrix} \tag{4.8}$$

is called the *period vector* of ω. For any pair of closed one-forms ω, χ on Σ one has

$$\int_\Sigma \omega \wedge \chi = \sum_{i=1}^{\hat{g}} \left(\int_{\alpha^i} \omega \int_{\beta_i} \chi - \int_{\beta_i} \omega \int_{\alpha^i} \chi \right) \ . \tag{4.9}$$

This is the *Riemann bilinear* relation for Riemann surfaces.

Denote the $\hat{g}$ linearly independent holomorphic one-forms on Σ by $\{\lambda_k\}$, $k \in \{1, \ldots, \hat{g}\}$. Define

$$e^i_k := \int_{\alpha^i} \lambda_k \quad , \quad h_{ik} := \int_{\beta_i} \lambda_k \ , \tag{4.10}$$

and from these the *period matrix*

$$\Pi_{ij} := h_{ik}(e^{-1})^k_j \ . \tag{4.11}$$

Inserting two holomorphic forms λ_i, λ_j for ω, χ in (4.9) the left-hand side vanishes, which tells us that the period matrix is symmetric. This is known as *Riemann's first relation*. Furthermore, using (4.9) with $\lambda_k, \bar{\lambda}_{\bar{l}}$ one finds *Riemann's second relation*,

$$\mathrm{Im}(\Pi_{ij}) > 0. \tag{4.12}$$

Riemann's relations are invariant under a symplectic change of the homology basis.

The moduli space of Riemann surfaces
Let $\mathcal{M}_{\hat{g}}$ denote the moduli space of complex structures on a genus $\hat{g}$ Riemann surface. As is reviewed in appendix B.1, infinitesimal changes of the complex structure of a manifold X are described by $H^1_{\bar{\partial}}(TX)$ and therefore this vector space is the tangent space to $\mathcal{M}_{\hat{g}}$ at the point corresponding to X. This is interesting because the dimension of $\mathcal{M}_{\hat{g}}$ coincides with the dimension of its tangent space and the latter can be computed explicitly (using the Grothendieck-Riemann-Roch formula, see [81]). The result is

$$\begin{aligned} \mathcal{M}_0 &= \{\text{point}\} \ , \\ \dim_{\mathbb{C}}\mathcal{M}_1 &= 1 \ , \\ \dim_{\mathbb{C}}\mathcal{M}_{\hat{\mathrm{g}}} &= 3\hat{g}-3 \quad \text{for } \hat{g} \geq 2 \ . \end{aligned} \tag{4.13}$$

One might consider the case in which one has additional marked points on the Riemann surface. The corresponding moduli space is denoted by $\mathcal{M}_{\hat{g},n}$ and its dimension is given by

$$\begin{aligned} \dim_{\mathbb{C}}\mathcal{M}_{0,\mathrm{n}} &= n-3 \quad \text{for } n \geq 3 \ , \\ \dim_{\mathbb{C}}\mathcal{M}_{1,\mathrm{n}} &= n \ , \\ \dim_{\mathbb{C}}\mathcal{M}_{\hat{\mathrm{g}}} &= 3\hat{g}-3+n \quad \text{for } \hat{g} \geq 2 \ . \end{aligned} \tag{4.14}$$

4.2 Properties of (local) Calabi-Yau manifolds

4.2.1 Aspects of compact Calabi-Yau manifolds

Our definition of a Calabi-Yau manifold is similar to the one of [86].

Definition 4.2 Let X be a compact complex manifold of complex dimension m and J the complex structure on X. A *Calabi-Yau manifold* is a triple (X, J, g), s.t. g is a Kähler metric on (X, J) with holonomy group $Hol(g) = SU(m)$.

A Calabi-Yau manifold of dimension m admits a nowhere vanishing, covariantly constant holomorphic $(m, 0)$-form Ω on X that is unique up to multiplication by a non-zero complex number.

Proposition 4.3 *Let (X, J, g) be a Calabi-Yau manifold, then g is Ricci-flat. Conversely, if (X, J, g) of dimension m is simply connected with a Ricci-flat Kähler metric, then its holonomy group is contained in $SU(m)$.*

Note that Ricci-flatness implies $c_1(TX) = 0$. The converse follows from Calabi's conjecture:

Proposition 4.4 *Let (X, J) be a compact complex manifold with $c_1(X) = [0] \in H^2(X;\mathbb{R})$. Then every Kähler class $[\omega]$ on X contains a unique Ricci-flat Kähler metric g.*

In our definition of a Calabi-Yau manifold we require the holonomy group to be precisely $SU(m)$ and not a proper subgroup. It can be shown that the first Betti-number of these manifolds vanishes, $b_1 = b^1 = 0$.

We are mainly interested in Calabi-Yau three-folds. The Hodge numbers of these can be shown to form the following *Hodge diamond*:

$$
\begin{array}{ccccccc}
 & & & h^{0,0} & & & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\
 & h^{3,1} & & h^{2,2} & & h^{1,3} & \\
 & & h^{3,2} & & h^{2,3} & & \\
 & & & h^{3,3} & & &
\end{array}
\quad = \quad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

The dimension of the homology group $H_3(X;\mathbb{Z})$ is $2h^{2,1}+2$ and one can always choose a symplectic basis[2] $\Gamma_{\alpha^I}, \Gamma_{\beta_J}$, $I, J \in \{0, \ldots, h^{2,1}\}$ with intersection matrix similar to the one in (4.4).

The period vector of the holomorphic form Ω is defined as

$$
\Pi(z) := \begin{pmatrix} \int_{\Gamma_{\alpha^I}} \Omega \\ \int_{\Gamma_{\beta_J}} \Omega \end{pmatrix} . \tag{4.15}
$$

Similar to the bilinear relation on Riemann surfaces one has for two closed three-forms Σ, Ξ on a (compact) Calabi-Yau manifold,

$$
\langle \Sigma, \Xi \rangle := \int_X \Sigma \wedge \Xi = \sum_I \left(\int_{\Gamma_{\alpha^I}} \Sigma \int_{\Gamma_{\beta_I}} \Xi - \int_{\Gamma_{\beta_I}} \Sigma \int_{\Gamma_{\alpha^I}} \Xi \right) . \tag{4.16}
$$

The moduli space of (compact) Calabi-Yau three-folds

This and the next subsection follow mainly the classic paper [34]. Let (X, J, Ω, g) be a Calabi-Yau manifold of complex dimension $m = 3$. We are interested in the moduli space $\mathcal{M}$, which we take to be the space of all Ricci-flat Kähler metrics on X. Note that in this definition of the moduli space it is implicit that the topology of the Calabi-Yau space is kept fixed. In particular the numbers $b_2 = h_{1,1}$ for the two-cycles and $b_3 = 2(h_{2,1} + 1)$ for the three-cycles are fixed once and for all.[3] Following [34] we start from the condition of Ricci-flatness, $\mathcal{R}_{AB}(g) = 0$, satisfied on every Calabi-Yau manifold. Here $A, B, \ldots$ label real coordinates on the Calabi-Yau X. In order to explore

[2] We use the letters (α^i, β_j), (A^i, B_j), (a^i, b_j), $\ldots$ to denote symplectic bases on Riemann surfaces and $(\Gamma_{\alpha^I}, \Gamma_{\beta_J})$, $(\Gamma_{A^I}, \Gamma_{B_J})$, $(\Gamma_{a^I}, \Gamma_{b_J})$, $\ldots$ for symplectic bases of three-cycles on Calabi-Yau three-folds. Also, the index i runs from 1 to $\hat{g}$, whereas I runs from 0 to $h^{2,1}$.

[3] In fact this condition can be relaxed if one allows for singularities. Then moduli spaces of Calabi-Yau manifolds of different topology can be glued together consistently. See [33] and [70] for an illuminating discussion.

the space of metrics we simply deform the original metric and require Ricci-flatness to be maintained,

$$\mathcal{R}_{AB}(g+\delta g)=0\ . \tag{4.17}$$

Of course, starting from one "background" metric and deforming it only explores the moduli space in a neighbourhood of the original metric and we only find a local description of $\mathcal{M}$. Its global structure is in general very hard to describe. After some algebra (4.17) turns into the Lichnerowicz equation

$$\nabla^C\nabla_C\delta g_{AB}+2R_{A\ B}^{\ D\ E}\delta g_{DE}=0\ . \tag{4.18}$$

Next we introduce complex coordinates x^μ on X, with $\mu,\nu,\ldots=1,2,3$. Then there are two possible deformations of the metric, namely $\delta g_{\mu\nu}$ or $\delta g_{\mu\bar{\nu}}$. Plugging these into (4.18) leads to two independent equations, one for $\delta g_{\mu\nu}$ and one for $\delta g_{\mu\bar{\nu}}$ and therefore the two types of deformations can be studied independently. To each variation of the metric of mixed type one can associate the real $(1,1)$-form $i\delta g_{\mu\bar{\nu}}\mathrm{d}x^\mu\wedge\mathrm{d}\bar{x}^{\bar{\nu}}$, which can be shown to be harmonic if and only if $\delta g_{\mu\bar{\nu}}$ satisfies the Lichnerowicz equation. A variation of pure type can be associated to the (2,1)-form $\Omega_{\kappa\lambda}{}^{\bar{\nu}}\delta g_{\bar{\mu}\bar{\nu}}\mathrm{d}x^\kappa\wedge\mathrm{d}x^\lambda\wedge\mathrm{d}\bar{x}^{\bar{\mu}}$, which also is harmonic if and only if $\delta g_{\bar{\mu}\bar{\nu}}$ satisfies (4.18). This tells us that the allowed transformations of the metric are in one-to-one correspondence with $H^{(1,1)}(X)$ and $H^{(2,1)}(X)$. The interpretation of the mixed deformations $\delta g_{\mu\bar{\nu}}$ is rather straightforward as they lead to a new Kähler form,

$$\begin{aligned}\tilde{K} &= i\tilde{g}_{\mu\bar{\nu}}\mathrm{d}x^\mu\wedge\mathrm{d}\bar{x}^{\bar{\nu}}=i(g_{\mu\bar{\nu}}+\delta g_{\mu\bar{\nu}})\mathrm{d}x^\mu\wedge\mathrm{d}\bar{x}^{\bar{\nu}}\\ &= K+i\delta g_{\mu\bar{\nu}}\mathrm{d}x^\mu\wedge\mathrm{d}\bar{x}^{\bar{\nu}}=K+\delta K\ .\end{aligned} \tag{4.19}$$

The variation $\delta g_{\mu\nu}$ on the other hand is related to a variation of the complex structure. To see this note that $\tilde{g}_{AB}=g_{AB}+\delta g_{AB}$ is a Kähler metric close to the original one. Then there must exist a coordinate system in which the pure components of the metric $\tilde{g}_{AB}$ vanish. Under a change of coordinates $x^A\to x'^A:=x^A+f^A(x)=h^A(x)$ we have

$$\begin{aligned}\tilde{g}'_{AB} &= \left(\frac{\partial h}{\partial x}\right)^{-1C}_{A}\left(\frac{\partial h}{\partial x}\right)^{-1D}_{B}\tilde{g}_{CD}\\ &= g_{AB}+\delta g_{AB}-(\partial_A f^C)g_{CB}-(\partial_B f^D)g_{AD}\ .\end{aligned} \tag{4.20}$$

We start from a mixed metric $g_{\mu\bar{\nu}}$ and add a pure deformation $\delta g_{\bar{\mu}\bar{\nu}}$. The resulting metric can be written as a mixed metric in some coordinate system, we only have to chose f s.t.

$$\delta g_{\bar{\mu}\bar{\nu}}-(\partial_{\bar{\mu}}f^\rho)g_{\rho\bar{\nu}}-(\partial_{\bar{\nu}}f^\rho)g_{\rho\bar{\mu}}=0\ . \tag{4.21}$$

But this means that f cannot be chosen to be holomorphic and thus we change the complex structure. Note that the fact that the deformations of complex structure are characterised by $H^{2,1}(X)$ is consistent with the discussion in appendix B.1 since $H^{2,1}(X)\cong H^1_{\bar{\partial}}(TX)$.

Next let us define a metric on the space of all Ricci-flat Kähler metrics,

$$\mathrm{d}s^2=\frac{1}{4\mathrm{vol(X)}}\int_X g^{AC}g^{BD}(\delta g_{AB}\delta g_{CD})\sqrt{g}\ \mathrm{d}^6x\ . \tag{4.22}$$

In complex coordinates one finds

$$\mathrm{d}s^2 = \frac{1}{2\mathrm{vol(X)}} \int_X g^{\mu\bar{\kappa}} g^{\nu\bar{\lambda}} \left[\delta g_{\mu\nu}\delta g_{\bar{\kappa}\bar{\lambda}} + \delta g_{\mu\bar{\lambda}}\delta g_{\nu\bar{\kappa}}\right] \sqrt{g}\ \mathrm{d}^6 x\ . \tag{4.23}$$

Interestingly, this metric is block-diagonal with separate blocks corresponding to variations of the complex and Kähler structure.

Complex structure moduli

Starting from one point in the space of all Ricci-flat metrics on a Calabi-Yau manifold X, we now want to study the space of those metrics that can be reached from that point by deforming the complex structure of the manifold, while keeping the Kähler form fixed. The space of these metrics is the moduli space of complex structures and it is denoted $\mathcal{M}_{cs}$. Set

$$\chi_i := \frac{1}{2}\chi_{i\mu\nu\bar{\lambda}} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}\bar{x}^{\bar{\lambda}} \quad \text{with} \quad \chi_{i\mu\nu\bar{\lambda}} := -\frac{1}{2}\Omega_{\mu\nu}{}^{\bar{\rho}} \frac{\partial g_{\bar{\lambda}\bar{\rho}}}{\partial z^i}\ , \tag{4.24}$$

where the z^i for $i \in \{1, \ldots, h_{2,1}\}$ are the parameters for the complex structure deformation, i.e. they are coordinates on $\mathcal{M}_{cs}$. Clearly, χ_i is a $(2,1)$-form $\forall i$. One finds

$$\bar{\Omega}_{\bar{\rho}}{}^{\mu\nu}\chi_{i\mu\nu\bar{\lambda}} = -||\Omega||^2 \frac{\partial g_{\bar{\rho}\bar{\lambda}}}{\partial z^i}\ , \tag{4.25}$$

where we used that $\Omega_{\mu\nu\lambda} = \Omega_{123}\tilde{\epsilon}_{\mu\nu\lambda}$ with $\tilde{\epsilon}$ a tensor density, and $||\Omega||^2 := \sqrt{g}^{-1}\Omega_{123}\bar{\Omega}_{123} = \frac{1}{3!}\Omega_{\mu\nu\rho}\bar{\Omega}^{\mu\nu\rho}$. This gives

$$\delta g_{\bar{\rho}\bar{\lambda}} = -\frac{1}{||\Omega||^2}\bar{\Omega}_{\bar{\rho}}{}^{\mu\nu}\chi_{i\mu\nu\bar{\lambda}}\delta z^i\ . \tag{4.26}$$

We saw that the metric on moduli space can be written in block diagonal form. At the moment we are interested in the complex structure only and we write a metric on $\mathcal{M}_{cs}$

$$2G^{(cs)}_{i\bar{j}}\delta z^i \delta\bar{z}^{\bar{j}} := \frac{1}{2\mathrm{vol(X)}} \int_X g^{\kappa\bar{\nu}} g^{\mu\bar{\lambda}} \delta g_{\kappa\mu}\delta g_{\bar{\lambda}\bar{\nu}} \sqrt{g}\ \mathrm{d}^6 x\ . \tag{4.27}$$

Using (4.26) we find

$$G^{(cs)}_{i\bar{j}} = -\frac{\int_X \chi_i \wedge \bar{\chi}_{\bar{j}}}{\int_X \Omega \wedge \bar{\Omega}}\ , \tag{4.28}$$

where we used that $||\Omega||^2$ is a constant on X, which follows from the fact that Ω is covariantly constant, and $\int_X \Omega \wedge \bar{\Omega} = ||\Omega||^2 \mathrm{vol(X)}$. The factor of 2 multiplying $G^{(cs)}_{i\bar{j}}$ was chosen to make this formula simple. To proceed we need the important formula

$$\frac{\partial\Omega}{\partial z^i} = k_i\Omega + \chi_i\ , \tag{4.29}$$

where k_i may depend on the z^j but not on the coordinates of X. See for example [34] for a proof. Using (4.29) it is easy to show that

$$G^{(cs)}_{i\bar{j}} = -\frac{\partial}{\partial z^i}\frac{\partial}{\partial\bar{z}^{\bar{j}}} \log\left(i\int_X \Omega\wedge\bar{\Omega}\right)\ , \tag{4.30}$$

which tells us that the metric (4.27) on the moduli space of complex structures $\mathcal{M}_{cs}$ is Kähler with Kähler potential

$$K = -\log\left(i\int_X \Omega\wedge\bar{\Omega}\right) . \tag{4.31}$$

If we differentiate this equation with respect to z^i we can use (4.29) to find that $k_i = -\frac{\partial}{\partial z^i}K$.

Next we consider the *Hodge bundle* $\mathcal{H}$, which is nothing but the cohomology bundle over $\mathcal{M}_{cs}$ s.t. at a given point $z\in\mathcal{M}_{cs}$ the fiber is given by $H^3(X_z;\mathbb{C})$, where X_z is the manifold X equipped with the complex structure $J(z)$ determined by the point $z\in\mathcal{M}_{cs}$. This bundle comes with a natural flat connection which is known as the *Gauss-Manin connection*. Let us explain this in more detail. One defines a Hermitian metric on the Hodge bundle as

$$(\eta,\theta) := i\int_X \eta\wedge\theta \quad \text{for } \eta,\ \theta\in H^3(X;\mathbb{C}) . \tag{4.32}$$

This allows us to define a symplectic basis of real integer three-forms $\omega_I,\eta^J\in H^3(X;\mathbb{C})$, $I,J\in\{0,1,\dots h^{2,1}\}$ with the property

$$(\omega_I,\eta^J) = -(\eta^J,\omega_I) = \delta_I^J , \tag{4.33}$$

which is unique up to a symplectic transformation. Dual to the basis of $H^3(X;\mathbb{Z})\subset H^3(X;\mathbb{C})$ there is a symplectic basis of three-cycles $\{\Gamma_{\alpha^I},\Gamma_{\beta_J}\}\in H_3(X;\mathbb{Z})$, s.t.

$$\int_{\Gamma_{\alpha^I}}\omega_J = \delta_J^I \quad , \quad \int_{\Gamma_{\beta_I}}\eta^J = \delta_I^J , \tag{4.34}$$

and all other combinations vanish. Clearly, the corresponding intersection matrix is

$$\begin{aligned} &\Gamma_{\alpha^I}\cap\Gamma_{\alpha^j} = \emptyset \quad , \quad \Gamma_{\beta_J}\cap\Gamma_{\beta_J} = \emptyset , \\ &\Gamma_{\alpha^I}\cap\Gamma_{\beta_J} = -\Gamma_{\beta_j}\cap\Gamma_{\alpha^i} = \delta_j^i , \end{aligned} \tag{4.35}$$

see for example [71] for a detailed treatment of these issues. Note that the Hermitian metric is defined on every fibre of the Hodge bundle, so we actually find three-forms and three-cycles at every point, $\omega_I(z),\ \eta^J(z),\ \Gamma_{\alpha^I}(z),\ \Gamma_{\beta^J}(z)$. Since the topology of X does not change if we move in the moduli space one can identify the set of basis cycles at one point p_1 in moduli space with the set of basis cycles at another point p_2. To do so one takes a path connecting p_1 and p_2 and identifies a cycle at p_1 with the cycle at p_2 which arises from the cycle at p_1 by following the chosen path.[4] For a detailed

[4]This might sound somewhat complicated but it is in fact very easy. Consider as an example a torus T^2 with its standard cycles α,β. If we now change the size (or the shape) of the torus we will find a new set of cycles α',β'. These can in principle be chosen arbitrarily. However, there is a natural choice which we want to identify with α,β, namely those that arise from α,β by performing the scaling.

explanation in the context of singularity theory see [15]. Note that this identification is unique if the space is simply connected. The corresponding connection is the Gauss-Manin connection. Clearly, this connection is flat, since on a simply connected domain of $\mathcal{M}_{cs}$ the identification procedure does not depend on the chosen path. If the domain is not simply connected going around a non-contractible closed path in moduli space will lead to a monodromy transformation of the cycles. Since we found that there is a natural way to identify basis elements of $H_3(X_z;\mathbb{Z})$ at different points $z \in \mathcal{M}_{cs}$ we can also identify the corresponding dual elements $\omega_I(z), \eta^J(z)$. Then, by definition, a section σ of the cohomology bundle is covariantly constant with respect to the Gauss-Manin connection if, when expressed in terms of the basis elements $\omega_I(z), \eta^J(z)$, its coefficients do not change if we move around in $\mathcal{M}_{cs}$,

$$\sigma(z) = \sum_{I=0}^{h^{2,1}(X)} c^I \omega_I(z) + \sum_{J=0}^{h^{2,1}(X)} d_J \eta^J(z) \quad \forall\, z \in \mathcal{M}_{cs} \ . \tag{4.36}$$

In particular, the basis elements ω_I, η^J are covariantly constant. A holomorphic section ρ of the cohomology bundle is given by

$$\rho(z) = \sum_{I=0}^{h^{2,1}(X)} f^I(z)\omega_I(z) + \sum_{J=0}^{h^{2,1}(X)} g_J(z)\eta^J(z) \ , \tag{4.37}$$

where $f^I(z), g_J(z)$ are holomorphic functions on $\mathcal{M}_{cs}$.

If we move in the base space $\mathcal{M}_{cs}$ of the Hodge bundle $\mathcal{H}$ we change the complex structure on X and therefore we change the Hodge-decomposition of the fibre $H^3(X_z;\mathbb{C}) = \oplus_{k=0}^3 H^{(3-k,k)}(X_z)$. Thus studying the moduli space of Calabi-Yau manifolds amounts to studying the variation of the Hodge structure of the Hodge bundle. Consider the holomorphic $(3,0)$-form, defined on every fibre X_z. The set of all these forms defines a holomorphic section of the Hodge bundle. On a given fibre X_z the form Ω is only defined up to multiplication by a non-zero constant. The section Ω of the Hodge bundle is then defined only up to a multiplication of a nowhere vanishing holomorphic function $e^{f(z)}$ on $\mathcal{M}_{cs}$ which amounts to saying that one can multiply Ω by different constants on different fibres, as long as they vary holomorphically in z. It is interesting to see in which way this fact is related to the properties of the Kähler metric on $\mathcal{M}_{cs}$. The property that Ω is defined only up to multiplication of a holomorphic function,

$$\Omega \to \Omega' := e^{f(z)}\Omega \ , \tag{4.38}$$

implies

$$K \quad \to \quad \tilde{K} = K + f(z) + \bar{f}(\bar{z}) \ , \tag{4.39}$$

$$G^{(CS)}_{i\bar{j}} \quad \to \quad \tilde{G}^{(CS)}_{i\bar{j}} = G^{(CS)}_{i\bar{j}} \ . \tag{4.40}$$

So a change of $\Omega(z)$ can be understood as a Kähler transformation which leaves the metric on moduli space unchanged.

Next we express the holomorphic section Ω of the Hodge bundle as[5]

$$\Omega(z) = X^I(z)\omega_I + \mathcal{F}_J(z)\eta^J \ , \tag{4.41}$$

where we have

$$X^I(z) = \int_{\Gamma_{\alpha^I}} \Omega(z) \quad , \quad \mathcal{F}_J(z) = \int_{\Gamma_{\beta_J}} \Omega(z) \ . \tag{4.42}$$

Since $\Omega(z)$ is holomorphic both X^I and $\mathcal{F}_J$ are holomorphic functions of the coordinates z on moduli space. By definition for two points in moduli space, say z, z', the corresponding three-forms $\Omega(z), \Omega(z')$ are different. Since the bases $\omega^I(z), \eta_j(z)$ and $\omega^I(z'), \eta_j(z')$ at the two points are identified and used to compare forms at different points in moduli space it is the coefficients $X^I, \mathcal{F}_J$ that must change if we go from z to z'. In fact, we can take a subset of these, say the X^I to form coordinates on moduli space. Since the dimension of $\mathcal{M}_{cs}$ is $h^{2,1}$ but we have $h^{2,1}+1$ functions X^I these have to be homogeneous coordinates. Then the $\mathcal{F}_J$ can be expressed in terms of the X^I. Let us then take the X^I to be homogeneous coordinates on $\mathcal{M}_{cs}$ and apply the Riemann bilinear relation (4.16) to

$$\int \Omega \wedge \frac{\partial \Omega}{\partial X^I} = 0 \ . \tag{4.43}$$

This gives[6]

$$\mathcal{F}_I = X^J \frac{\partial}{\partial X^I}\mathcal{F}_J = \frac{1}{2}\partial_J(X^I \mathcal{F}_I) = \frac{\partial}{\partial X^I}\mathcal{F} \ , \tag{4.44}$$

where

$$\mathcal{F} := \frac{1}{2} X^I \mathcal{F}_I \ . \tag{4.45}$$

So the $\mathcal{F}_I$ are derivatives of a function $\mathcal{F}(X)$ which is homogeneous of degree 2. $\mathcal{F}$ is called the *prepotential*. This nomenclature comes from the fact that the Kähler potential can itself be expressed in terms of $\mathcal{F}$,

$$K = -\log\left(i\int \Omega \wedge \bar{\Omega}\right) = -\log\left(i\sum_{I=0}^{h^{2,1}} \left(X^I \bar{\mathcal{F}}_I - \bar{X}^I \mathcal{F}_I\right)\right) \ . \tag{4.46}$$

We will have to say much more about this structure below.

To summarise, we found that the moduli space of complex structures of a Calabi-Yau manifold carries a Kähler metric with a Kähler potential that can be calculated from the geometry of the Calabi-Yau. A Kähler transformation can be understood as an irrelevant multiplication of the section $\Omega(z)$ by a nowhere vanishing holomorphic function. The coordinates of the moduli space can be obtained from integrals of $\Omega(z)$ over the Γ_{α^I}-cycles and the integrals over the corresponding Γ_{β_J}-cycles can then be shown to be derivatives of a holomorphic function $\mathcal{F}$ in the coordinates X, in terms of which the Kähler potential can be expressed. As explained in appendix C.1 these properties determine $\mathcal{M}_{cs}$ to be a special Kähler manifold.

[5] Since the ω_I, η_J on different fibres are identified we omit their z-dependence.

[6] To be more precise one should have defined $\tilde{\mathcal{F}}_J(z) := \int_{\Gamma_{\beta_J}} \Omega(z)$ and $\mathcal{F}_J(X) := \tilde{\mathcal{F}}_J(z(X))$.

Obviously the next step would be to analyse the structure of the moduli space of Kähler structures, $\mathcal{M}_{KS}$. Indeed, this has been studied in [34] and the result is that $\mathcal{M}_{KS}$ also is a special Kähler manifold, with a Kähler potential calculable from some prepotential. However, the prepotential now is no longer a simple integral in the geometry, but it receives instanton correction. Hence, in general it is very hard to calculate it explicitly. This is where results from mirror symmetry come in useful. Mirror symmetry states that Calabi-Yau manifolds come in pairs and that the Kähler structure prepotential can be calculated by evaluating the geometric integral on the mirror manifold and using what is known as the mirror map. Unfortunately, we cannot delve any further into this fascinating subject, but must refer the reader to [70] or [81].

4.2.2 Local Calabi-Yau manifolds

After an exposition of the properties of (compact) Calabi-Yau spaces we now turn to the spaces which are used to geometrically engineer the gauge theories that we want to study.

Definition 4.5 A *local Calabi-Yau manifold* is a non-compact Kähler manifold with vanishing first Chern-class.

Next we give a series of definitions which will ultimately lead us to an explicit local Calabi-Yau manifold. For completeness we start from the definition of $\mathbb{CP}^1$.

Definition 4.6 The *complex projective space* $\mathbb{CP}^1 \equiv \mathbb{P}^1$ is defined as

$$\mathbb{C}^2 \backslash \{0\} / \sim \ . \tag{4.47}$$

For $(z_1, z_2) \in \mathbb{C}^2 \backslash \{0\}$ the equivalence relation is defined as

$$(z_1, z_2) \sim (\lambda z_1, \lambda z_2) = \lambda(z_1, z_2) \tag{4.48}$$

for $\lambda \in \mathbb{C} \backslash \{0\}$. Note that this implies that $\mathbb{CP}^1$ is the space of lines through 0 in $\mathbb{C}^2$. We can introduce patches on $\mathbb{CP}^1$ as follows

$$\begin{aligned} U_1^{\mathbb{P}^1} &:= \{(z_1, z_2) \in \mathbb{C}^2 \backslash \{0\} : z_1 \neq 0, (z_1, z_2) \sim \lambda(z_1, z_2)\} \ , \\ U_2^{\mathbb{P}^1} &:= \{(z_1, z_2) \in \mathbb{C}^2 \backslash \{0\} : z_2 \neq 0, (z_1, z_2) \sim \lambda(z_1, z_2)\} \ , \end{aligned} \tag{4.49}$$

and on these patches we can introduce coordinates

$$\xi_1 := \frac{z_2}{z_1} \quad , \quad \xi_2 := \frac{z_1}{z_2} \ . \tag{4.50}$$

On the overlap $U_1 \cap U_2$ we have

$$\xi_2 = \frac{1}{\xi_1} \ , \tag{4.51}$$

and we find that $\mathbb{CP}^1$ is isomorphic to a Riemann sphere S^2.

Definition 4.7 The space $\mathcal{O}(n) \to \mathbb{CP}^1$ is a line bundle over $\mathbb{CP}^1$. We can define it in terms of charts

$$\begin{aligned} U_1 &:= \{(\xi_1, \Phi) : \xi_1 \in U_1^{\mathbb{P}^1} \cong \mathbb{C}, \Phi \in \mathbb{C}\} , \\ U_2 &:= \{(\xi_2, \Phi') : \xi_2 \in U_2^{\mathbb{P}^1} \cong \mathbb{C}, \Phi' \in \mathbb{C}\} , \end{aligned} \tag{4.52}$$

with

$$\xi_2 = \frac{1}{\xi_1} \quad , \quad \Phi' = \xi_1^{-n}\Phi \tag{4.53}$$

on $U_1 \cap U_2$.

Definition 4.8 Very similarly $\mathcal{O}(m) \oplus \mathcal{O}(n) \to \mathbb{CP}^1$ is a fibre bundle over $\mathbb{CP}^1$ where the fibre is a direct sum of two complex planes. We define it via coordinate charts and transition functions

$$\begin{aligned} U_1 &:= \{(\xi_1, \Phi_0, \Phi_1) : \xi_1 \in U_1^{\mathbb{P}^1}, \Phi_0 \in \mathbb{C}, \Phi_1 \in \mathbb{C}\} , \\ U_2 &:= \{(\xi_2, \Phi'_0, \Phi'_1) : \xi_2 \in U_2^{\mathbb{P}^1}, \Phi'_0 \in \mathbb{C}, \Phi'_1 \in \mathbb{C}\} , \end{aligned} \tag{4.54}$$

with

$$\xi_2 = \frac{1}{\xi_1} \quad , \quad \Phi'_0 = \xi_1^{-m}\Phi_0 \quad , \quad \Phi'_1 = \xi_1^{-n}\Phi_1 \quad \text{on } U_1 \cap U_2 . \tag{4.55}$$

These manifolds are interesting because of the following proposition, which is explained in [104] and [71].

Proposition 4.9 *The first Chern class of* $\mathcal{O}(m)\oplus\mathcal{O}(n) \to \mathbb{CP}^1$ *vanishes if* $m+n = -2$.

The conifold

Definition 4.10 The *conifold* C_0 is defined as $f^{-1}(0)$ with f given by

$$\begin{aligned} f : \mathbb{C}^4 &\to \mathbb{C} \\ (w_1, w_2, w_3, w_4) &\mapsto f(w_1, w_2, w_4, w_4) := w_1^2 + w_2^2 + w_3^2 + w_4^2 . \end{aligned} \tag{4.56}$$

In other words

$$C_0 := \{\vec{w} \in \mathbb{C}^4 : w_1^2 + w_2^2 + w_3^2 + w_4^2 = 0\} . \tag{4.57}$$

Setting $\Phi_0 = w_1 + iw_2$, $\Phi_1 = iw_3 - w_4$, $\Phi'_0 = iw_3 + w_4$ and $\Phi'_1 = w_1 - iw_2$ this reads

$$C_0 := \{(\Phi_0, \Phi'_0, \Phi_1, \Phi'_1)^\tau \in \mathbb{C}^4 : \Phi_0\Phi'_1 - \Phi'_0\Phi_1 = 0\} . \tag{4.58}$$

Clearly, f has a singularity at zero with singular value zero and, therefore C_0 is a singular manifold. To study the structure of C_0 in more detail we set $w_i = x_i + iy_i$. Then $f = 0$ reads

$$\vec{x}^2 - \vec{y}^2 = 0 \quad , \quad \vec{x} \cdot \vec{y} = 0 . \tag{4.59}$$

The first equation is $\vec{x}^2 = \frac{1}{2}r^2$ if $r^2 := \vec{x}^2 + \vec{y}^2$, so $\vec{x}$ lives on an S^3. $\vec{y}$ on the other hand is perpendicular to $\vec{x}$. For given r and x we have an S^2 of possible $\vec{y}$s and so for given r we have a fibre bundle of S^2 over S^3. However, there is no nontrivial fibration of S^2 over S^3 and we conclude that $f = 0$ is a cone over $S^3 \times S^2$. Fig. 4.3 gives an intuitive picture of the conifold, together with the two possible ways to resolve the singularity, namely its deformation and its small resolution, to which we will turn presently.

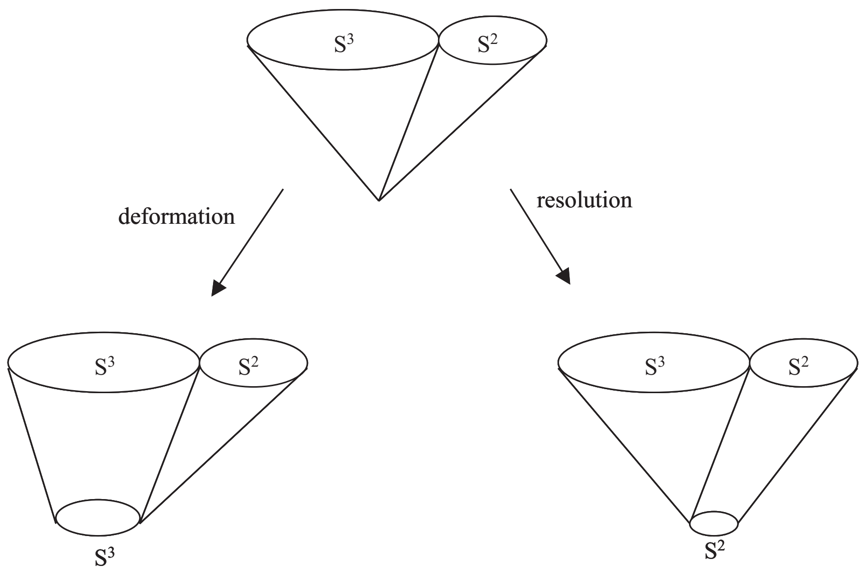

Figure 4.3: The conifold is a cone over $S^3 \times S^2$ with a conical singularity at its tip that can be smoothed out by a deformation or a small resolution.

The deformed conifold

Definition 4.11 The *deformed conifold* C_{def} is the set $f^{-1}(\mu)$ with $\mu \in \mathbb{R}_+$ and f as in (4.56).

In other words the deformed conifold is given by

$$C_{def} := \{\vec{w} \in \mathbb{C}^4 : w_1^2 + w_2^2 + w_3^2 + w_4^2 = \mu\} \tag{4.60}$$

or

$$C_{def} := \{(\Phi_0, \Phi_0', \Phi_1, \Phi_1')^\tau \in \mathbb{C}^4 : \Phi_0\Phi_1' - \Phi_0'\Phi_1 = \mu\} \ . \tag{4.61}$$

The above analysis of the structure of the conifold remains valid for fixed r, where again we have $S^3 \times S^2$. However, now we have an S^3 of minimal radius $\sqrt{\mu}$ that occurs for $\vec{y} = 0$. If we define $\vec{q} := \frac{1}{\sqrt{\mu + \vec{y}^2}}\vec{x}$ we find

$$\vec{q}^{\,2} = 1 \quad , \quad \vec{q} \cdot \vec{y} = 0 \ . \tag{4.62}$$

This shows that the deformed conifold is isomorphic to T^*S^3. Both the deformed conifold and the conifold are local Calabi-Yau. Interestingly, for these spaces this fact can be proven by writing down an explicit Ricci-flat Kähler metric [33].

The resolved conifold

The resolved conifold is given by the small resolution (see Def. B.1) of the set $f = 0$. We saw that the singular space can be characterised by

$$C := \{(\Phi_0, \Phi_0', \Phi_1, \Phi_1') \in \mathbb{C}^4 : \Phi_0\Phi_1' = \Phi_0'\Phi_1\} \ . \tag{4.63}$$

The small resolution of this space at $\vec{0} \in C \subset \mathbb{C}^4$ is in fact the space $\tilde{C} := C_{res} := \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$. To see this we have to construct a map $\pi : \tilde{C} \to C$ such that $\pi : \tilde{C}\backslash\pi^{-1}(\vec{0}) \to C\backslash\vec{0}$ is an isomorphism and $\pi^{-1}(\vec{0}) \cong \mathbb{CP}^1$. On the two patches of $\tilde{C}$ it is given by

$$\begin{array}{rcl} (\xi_1, \Phi_0, \Phi_1) & \overset{\pi}{\mapsto} & (\Phi_0, \xi_1\Phi_0, \Phi_1, \xi_1\Phi_1) \ , \\ (\xi_2, \Phi_0', \Phi_1') & \overset{\pi}{\mapsto} & (\xi_2\Phi_0', \Phi_0', \xi_2\Phi_1', \Phi_1') \ . \end{array} \tag{4.64}$$

A point in the overlap of the two charts in $\tilde{C}$ has to be mapped to the same point in C, which is indeed the case, since on the overlap we have $\xi_2 = \frac{1}{\xi_1}$. Note also that as long as Φ_0, Φ_1 do not vanish simultaneously this map is one to one. However, $(U_1^{\mathbb{P}^1}, 0, 0)$ and $(U_2^{\mathbb{P}^1}, 0, 0)$ are mapped to $(0, 0, 0, 0)$, s.t. $\pi^{-1}((0, 0, 0, 0)) \cong \mathbb{P}^1$. this proves that $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$ is indeed the small resolution of the conifold.

As in the case of the deformed conifold one can write down a Ricci-flat Kähler metric for the resolved conifold, see [33].

The resolved conifold and toric geometry

There is another very important description of the resolved conifold which appears in the context of linear sigma models and makes contact with toric geometry (see for example [147], [82], [81]). Let $\vec{z} = (z_1, z_2, z_3, z_4)^\tau \in \mathbb{C}^4$ and consider the space

$$C_{toric} := \{\vec{z} \in \mathbb{C}^4 : |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 = t\}/\sim \tag{4.65}$$

where the equivalence relation is generated by a $U(1)$ group that acts as

$$(z_1, z_2, z_3, z_4) \mapsto (e^{i\theta}z_1, e^{i\theta}z_2, e^{-i\theta}z_3, e^{-i\theta}z_4) \ . \tag{4.66}$$

This description appears naturally in the linear sigma model . In order to see that this is indeed isomorphic to C_{res} note that for $z_3 = z_4 = 0$ the space is isomorphic to $\mathbb{CP}^1$. Consider then the sets $U_i := \{\vec{z} \in C_{toric} : z_i \neq 0\}$, $i = 1, 2$. On U_1 we define the $U(1)$ invariant coordinates

$$\xi_1 := \frac{z_2}{z_1} \ , \ \Phi_0 := z_1z_3 \ , \ \Phi_1 := z_1z_4 \ , \tag{4.67}$$

and similarly for U_2,

$$\xi_2 := \frac{z_1}{z_2} \ , \ \Phi_0' := z_2z_3 \ , \ \Phi_1' := z_2z_4 \ , \tag{4.68}$$

Clearly, the ξ_i are the inhomogeneous coordinate on $U_i^{\mathbb{P}^1}$. On the overlap $U_1 \cap U_2$ we have

$$\xi_2 = \xi_1^{-1} \ , \ \Phi_0' = \xi_1 \Phi_0 \ , \ \Phi_1' = \xi_1 \Phi_1 \ , \tag{4.69}$$

which are the defining equations of C_{res}. Therefore, indeed, $C_{res} \cong C_{toric}$.

From this description of the resolved conifold we can now understand it as a T^3 fibration over (part of) $\mathbb{R}_+^3$ parameterised by $|z_1|^2$, $|z_3|^2$, $|z_4|^2$. Because of the defining equation $|z_2|^2 = t - |z_1|^2 + |z_3|^2 + |z_4|^2$ the base cannot consist of the entire $\mathbb{R}_+^3$. For example for $z_3 = z_4 = 0$ and $|z_1|^2 > t$ this equation has no solution. In fact, the boundary of the base is given by the hypersurfaces $|z_1|^2 = 0$, $|z_3|^2 = 0$, $|z_4|^2 = 0$ and $|z_2|^2 = 0$, see Fig. 4.4. The T^3 of the fiber is given by the phases of all the z_i

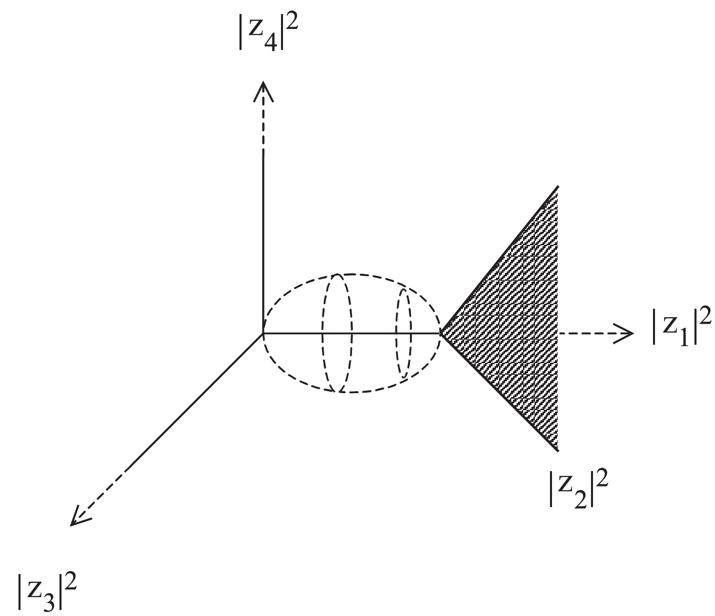

Figure 4.4: The part of $\mathbb{R}_+^3$ that is the base of the T^3 fibration of the resolved conifold is bounded by the surfaces $|z_1|^2 = 0$, $|z_3|^2 = 0$, $|z_4|^2 = 0$ and $|z_2|^2 = 0$. If k of these equations are satisfied simultaneously, k of the S^1s in T^3 shrinks to zero size. In particular, at $|z_3|^2 = |z_4|^2 = 0$ one has a single S^1 in the fibre that shrinks at $|z_1|^2 = 0$ and $|z_1|^2 = t$. The set of these S^1 form the $\mathbb{P}^1$ in C_{res}.

modulo the $U(1)$ transformation. The singularity locus of this fibration is then easily determined. In fact on every hypersurface $|z_i|^2 = 0$ the corresponding S^1 shrinks to zero size and the fibre consists of a T^2 only. At the loci where two of these surfaces intersect two S^1s shrink and we are left with an S^1. Finally, there are two points where three hypersurfaces intersect and the fibre degenerates to a point. This happens at $|z_3|^2 = |z_4|^2 = 0$ and $|z_1|^2 = 0$ or $|z_1|^2 = t$. If we follow the $|z_1|^2$-axis from 0 to t an S^1 opens up and shrinks again to zero. The set of these cycles form a sphere $S^2 \cong \mathbb{P}^1$, which is the $\mathbb{P}^1$ in C_{res}. For a detailed description of these circle fibrations see for example [100].

More general local Calabi-Yau manifolds

There is a set of more general local Calabi-Yau manifolds that was first constructed in [61] and which appeared in the physics literature in [87] and [27]. One starts from the bundle $\mathcal{O}(-2) \oplus \mathcal{O}(0) \to \mathbb{P}^1$ which is local Calabi-Yau. To make the discussion clear we once again write down the charts and the transition functions,

$$\begin{aligned} U_1 &:= \{(\xi_1, \Phi_0, \Phi_1) : \xi_1 \in U_1^{\mathbb{P}^1}, \Phi_0 \in \mathbb{C}, \Phi_1 \in \mathbb{C}\} \\ U_2 &:= \{(\xi_2, \Phi_0', \Phi_1') : \xi_1 \in U_2^{\mathbb{P}^1}, \Phi_0' \in \mathbb{C}, \Phi_1' \in \mathbb{C}\} \\ \text{with} \quad \xi_2 &= \frac{1}{\xi_1} \quad , \quad \Phi_0' = \xi_1^2 \Phi_0 \quad , \quad \Phi_1' = \Phi_1 \quad \text{on} \quad U_1 \cap U_2 \ . \end{aligned} \tag{4.70}$$

To get an intuitive picture of the structure of this space, we take $\Phi_0 = \Phi_0' = 0$ and fix $\Phi_1 = \Phi_1'$ arbitrarily. Then we can "walk around" in the ξ_i direction "consistently", i.e. we can change ξ_i without having to change the fixed values of Φ_0, Φ_1.

Next we consider a space X_{res} with coordinate patches U_1, U_2 as above but with transition functions

$$\xi_2 = \frac{1}{\xi_1} \quad , \quad \Phi_0' = \xi_1^2 \Phi_0 + W'(\Phi_1)\xi_1 \quad , \quad \Phi_1' = \Phi_1 \quad \text{on } U_1 \cap U_2 \ . \tag{4.71}$$

Here W is a polynomial of degree $n+1$. To see that the structure of this space is very different from the one of $\mathcal{O}(-2) \oplus \mathcal{O}(0) \to \mathbb{CP}^1$ we set $\Phi_0 = \Phi_0' = 0$ and fix Φ_1 arbitrarily, as before. Note that now ξ_1 is fixed and changing ξ_1 amounts to changing Φ_0'. Only for those specific values of Φ_1 for which $W'(\Phi_1) = 0$ can we consistently "walk" in the ξ_1-direction. Mathematically this means that the space that is deformed by a polynomial W contains n different $\mathbb{CP}^1$s.

We have seen that the "blow-down" of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$ is given by the conifold C. Now we are interested in the blow-down geometry of our deformed space, i.e. the geometry where the size of the $\mathbb{CP}^1$s is taken to zero. We claim that X_{res} can be obtained from a small resolution of all the singularities of the space

$$X := \{(\Phi_0, \Phi_0', \Phi_1, z) \in \mathbb{C}^4 : 4\Phi_0\Phi_0' + z^2 + W'(\Phi_1)^2 = 0\} \ . \tag{4.72}$$

As to prove that X_{res} is the small resolution of X we have to find a map $\pi : X_{res} \to X$. It is given by

$$\begin{aligned} (\xi_1, \Phi_0, \Phi_1) &\overset{\pi}{\mapsto} (\Phi_0, \xi_1^2\Phi_0 + W'(\Phi_1)\xi_1, \Phi_1, i(2\xi_1\Phi_0 + W'(\Phi_1)))\ , \\ (\xi_2, \Phi_0', \Phi_1') &\overset{\pi}{\mapsto} (\xi_2^2\Phi_0' - \xi_2 W'(\Phi_1'), \Phi_0', \Phi_1', i(2\xi_2\Phi_0' - W'(\Phi_1')))\ . \end{aligned}$$

It is easy to check that

- $\pi(U_i) \subset X$,
- on $U_1 \cap U_2$ the two maps map to the same point ,
- the map is one to one as long as $W'(\Phi_1) \neq 0$,
- for $\Phi_1 = \Phi_1'$ s.t. $W'(\Phi_1) = 0$ one finds that $(U_1^{\mathbb{P}^1}, 0, \Phi_1)$ and $(U_2^{\mathbb{P}^1}, 0, \Phi_1')$ are mapped to $(0, 0, \Phi_1, 0)$, i.e. $\pi^{-1}((0,0,\Phi_1,0)) \cong \mathbb{CP}^1 \ \forall \Phi_1$ s.t. $W'(\Phi_1) = 0$. This shows that X_{res} can indeed be understood as the small resolution of the singularities in X.

Changing coordinates $x = \Phi_1,\ v = \Phi_0 + \Phi_0',\ w = i(\Phi_0' - \Phi_0)$ the expression (4.72) for X can be rewritten as

$$X = \{(v, w, x, z) \in \mathbb{C}^4 : W'(x)^2 + v^2 + w^2 + z^2 = 0\} \ . \tag{4.73}$$

Finally, we note that it is easy to find the deformation of the singularities of (4.73). If $f_0(x)$ is a polynomial of degree $n-1$ then

$$\begin{aligned} X_{def} &:= \{(v, w, x, z) \in \mathbb{C}^4 : F(x, v, w, z) = 0\} \\ \text{with} \quad F(x, v, w, z) &:= W'(x)^2 + f_0(x) + v^2 + w^2 + z^2 \end{aligned} \tag{4.74}$$

is the space where (for generic coefficients of f_0) all the singularities are deformed. The spaces $X_{res},\ X$ and X_{def} are the local Calabi-Yau manifolds that will be used in order to geometrically engineer the gauge theories we are interested in. Going from the resolved space X_{res} through the singular space to the deformed one X_{def} is known as a *geometric transition*.

4.2.3 Period integrals on local Calabi-Yau manifolds and Riemann surfaces

We mentioned already in the introduction that one building block that is necessary to obtain the effective superpotential (2.14) of gauge theories is given by the integrals of the holomorphic $(3,0)$-form Ω, which exists on any (local) Calabi-Yau manifold, over all the (relative) three-cycles in the manifold. Here we will analyse these integrals on the space X_{def}, and review how they map to integrals on a Riemann surface, which is closely related to the local Calabi-Yau manifold we are considering.

Let us first concentrate on the definition of Ω. X_{def} is given by a (non-singular) hypersurface in $\mathbb{C}^4$. Clearly, on $\mathbb{C}^4$ there is a preferred holomorphic $(4,0)$-form, namely $\mathrm{d}x \wedge \mathrm{d}v \wedge \mathrm{d}w \wedge \mathrm{d}z$. Since X_{def} is defined by $F = 0$, where F is a holomorphic function in the x, v, w, z, the $(4,0)$-form on $\mathbb{C}^4$ induces a natural holomorphic $(3,0)$-form on X_{def}. To see this note that $\mathrm{d}F$ is perpendicular to the hypersurface $F = 0$. Then there is a unique holomorphic $(3,0)$-form on $F = 0$, such that $\mathrm{d}x \wedge \mathrm{d}v \wedge \mathrm{d}w \wedge \mathrm{d}z = \Omega \wedge \mathrm{d}F$. If $z \neq 0$ it can be written as

$$\Omega = \frac{\mathrm{d}x \wedge \mathrm{d}v \wedge \mathrm{d}w \wedge \mathrm{d}z}{\mathrm{d}F} = \frac{\mathrm{d}x \wedge \mathrm{d}v \wedge \mathrm{d}w}{2z} \ , \tag{4.75}$$

where z is a solution of $F = 0$. Turning to the three-cycles we note that, because of the simple dependence of the surface (4.74) on v, w and z every three-cycle of X_{def} can be understood as a fibration of a two-sphere over a line segment in the hyperelliptic Riemann surface Σ,

$$y^2 = W'(x)^2 + f_0(x) =: \prod_{i=1}^{n} (x - a_i^+)(x - a_i^-) \ , \tag{4.76}$$

of genus $\hat{g} = n-1$. This was first realised in [92] in a slightly different context, see [96] for a review. As explained above, this surface can be understood as two complex planes glued together along cuts running between a_i^- and a_i^+. Following the conventions of [P5] y_0, which is the branch of the Riemann surface with $y_0 \sim W'(x)$ for $|x| \to \infty$, is defined on the upper sheet and $y_1 = -y_0$ on the lower one. For compact three-cycles the line segment connects two of the branch points of the curve and the volume of the S^2-fibre depends on the position on the base line segment. At the end points of the segment one has $y^2 = 0$ and the volume of the sphere shrinks to zero size. Non-compact three-cycles on the other hand are fibrations of S^2 over a half-line that runs from one of the branch points to infinity on the Riemann surface. Integration over the fibre is elementary and gives

$$\int_{S^2} \Omega = \pm 2\pi i \; y(x)\mathrm{d}x \; , \tag{4.77}$$

(the sign ambiguity will be fixed momentarily) and thus the integral of the holomorphic Ω over a three-cycle is reduced to an integral of $\pm 2\pi i y \mathrm{d}x$ over a line segment in Σ. Clearly, the integrals over line segments that connect two branch points can be rewritten in terms of integrals over compact cycles on the Riemann surface, whereas the integrals over non-compact three-cycles can be expressed as integrals over a line that links the two infinities on the two complex sheets. In fact, the one-form

$$\zeta := y\mathrm{d}x \tag{4.78}$$

is meromorphic and diverges at infinity (poles of order $n+2$) on the two sheets and therefore it is well-defined only on the Riemann surface with the two infinities Q and Q' removed. Then, we are naturally led to consider the relative homology $H_1(\Sigma, \{Q, Q'\})$, which we encountered already when we discussed Riemann surfaces in section 4.1. To summarise, one ends up with a one-to-one correspondence between the (compact and non-compact) three-cycles in (4.74) and $H_1(\Sigma, \{Q, Q'\})$. Referring to our choice of bases $\{A^i, B_j\}$ respectively $\{\alpha^i, \beta_j\}$ for $H_1(\Sigma, \{Q, Q'\})$, as defined in Figs. 4.2 and 4.1, we define $\Gamma_{A^i}, \Gamma_{B_j}$ to be the S^2-fibrations over A^i, B_j, and $\Gamma_{\alpha^i}, \Gamma_{\beta_j}$ are S^2-fibrations over α^i, β_j. So the problem effectively reduces to calculating the integrals[7]

$$\int_{\Gamma_\gamma} \Omega = -i\pi \int_\gamma \zeta \quad \text{for} \quad \gamma \in \{\alpha^i, \beta_j, \hat{\alpha}, \hat{\beta}\} \; . \tag{4.79}$$

As we will see in the next chapter, these integrals can actually be calculated from a holomorphic matrix model.

As mentioned already, one expects new features to be contained in the integral $\int_{\hat{\beta}} \zeta$, where $\hat{\beta}$ runs from Q' on the lower sheet to Q on the upper one. Indeed, it is easy to see that this integral is divergent. It will be part of our task to understand and properly

[7] The sign ambiguity of (4.77) has now been fixed, since we have made specific choices for the orientation of the cycles. Furthermore, we use the (standard) convention that the cut of $\sqrt{x}$ is along the negative real axis of the complex x-plane. Also, on the right-hand side we used that the integral of ζ over the line segment is $\frac{1}{2}$ times the integral over a closed cycle γ.

treat this divergence. As usual, the integral will be regulated and one has to make sure that physical quantities do not depend on the regulator and remain finite once the regulator is removed. In the literature this is achieved by simply discarding the divergent part. Here we want to give a more intrinsic geometric prescription that will be similar to standard procedures in relative cohomology. To render the integral finite we simply cut out two "small" discs around the points Q, Q'. If x, x' are coordinates on the upper and lower sheet respectively, one only considers the domains $|x| \leq \Lambda_0$, $|x'| \leq \Lambda_0$, $\Lambda_0 \in \mathbb{R}$. Furthermore, we take the cycle $\hat{\beta}$ to run from the point Λ_0' on the real axis of the lower sheet to Λ_0 on the real axis of the upper sheet. (Actually one could take Λ_0 and Λ_0' to be complex. We will come back to this point later on.)

Chapter 5

Holomorphic Matrix Models and Special Geometry

After having collected some relevant background material let us now come back to the main line of our arguments. Our principal goal is to determine the effective superpotential (in the Veneziano-Yankielowicz sense, see section 3.3) of super Yang-Mills theory coupled to a chiral superfield in the adjoint representation with tree-level superpotential

$$W(\Phi) = \sum_{k=1}^{n+1} \frac{g_k}{k} \operatorname{tr} \Phi^k + g_0 \ . \tag{5.1}$$

We mentioned in the introduction that this theory can be geometrically engineered from type IIB string theory on the local Calabi-Yau manifolds X_{res} studied in section 4.2.2. Furthermore, as we will review below, Cachazo, Intriligator and Vafa claim that the effective superpotential of this theory can be calculated from integrals of Ω over all the three-cycles in the local Calabi-Yau X_{def}, which is obtained from X_{res} through a geometric transition.

We reviewed the structure of the moduli space of a *compact* Calabi-Yau manifold X, and we found the special geometry relations

$$\begin{aligned} X^I &= \int_{\Gamma_{A^I}} \Omega, \\ \mathcal{F}_I \equiv \frac{\partial \mathcal{F}}{\partial X^I} &= \int_{\Gamma_{B_I}} \Omega \ , \end{aligned} \tag{5.2}$$

where Ω is the unique holomorphic $(3,0)$-form on X, and $\{\Gamma_{A^I}, \Gamma_{B_J}\}$ is a symplectic basis of $H_3(X)$. On Riemann surfaces similar relations hold.

An obvious and important question to ask is whether we can find special geometry relations on the *non-compact* manifolds X_{def}, which would then be relevant for the computation of the effective superpotential. We already started to calculate the integrals of Ω over the three-cycles in section 4.2.3 and we found that they map to integrals of a meromorphic form on a Riemann surface. It is immediately clear that the naive

special geometry relations have to be modified, since we have at least one integral over a non-compact cycle $\Gamma_{\hat{\beta}}$, which is divergent. This can be remedied by introducing a cut-off Λ_0, but then the integral over the regulated cycle depends on the cut-off. The question we want to address in this chapter is how to evaluate the integrals of $\zeta = y\mathrm{d}x$ of Eq. (4.79) on the hyperelliptic Riemann surface (4.76). Furthermore, we derive a set of equations for these integrals on the Riemann surface which is similar to the special geometry relations (5.2), but which contain the cut-off Λ_0. Finally, a clear cut interpretation of the function $\mathcal{F}$ that appears in these relations is given. It turns out to be nothing but the free energy of a holomorphic matrix model at genus zero. For this reason we will spend some time explaining the holomorphic matrix model.

5.1 The holomorphic matrix model

The fact that the holomorphic matric model is relevant in this context was first discovered by Dijkgraaf and Vafa in [43], who noticed that the open topological B-model on X_{res} is related to a holomorphic matrix model with W as its potential. Then, in [45] they explored how the matrix model can be used to evaluate the effective superpotential of a quantum field theory. A general reference for matrix models is [59], particularly important for us are the results of [25]. Although similar to the Hermitean matrix model, the holomorphic matrix model has been studied only recently. In [95] Lazaroiu described many of its intriguing features, see also [91]. The subtleties of the saddle point expansion in this model, as well as some aspects of the special geometry relations were first studied in our work [P5].

5.1.1 The partition function and convergence properties

We begin by defining the partition function of the *holomorphic* one-matrix model following [95]. In order to do so, one chooses a smooth path $\gamma : \mathbb{R} \to \mathbb{C}$ without self-intersection, such that $\dot{\gamma}(u) \neq 0 \ \forall u \in \mathbb{R}$ and $|\gamma(u)| \to \infty$ for $u \to \pm\infty$. Consider the ensemble $\Gamma(\gamma)$ of[1] $\hat{N} \times \hat{N}$ complex matrices M with spectrum $\mathrm{spec}(M) = \{\lambda_1, \ldots \lambda_{\hat{N}}\}$ in[2] γ and distinct eigenvalues,

$$\Gamma(\gamma) := \{M \in \mathbb{C}^{\hat{N}\times\hat{N}} : \mathrm{spec}(M) \subset \gamma,\ \text{all } \lambda_m \text{ distinct}\} \ . \tag{5.3}$$

The holomorphic measure on $\mathbb{C}^{\hat{N}\times\hat{N}}$ is just $\mathrm{d}M \equiv \wedge_{p,q}\mathrm{d}M_{pq}$ with some appropriate sign convention [95]. The potential is given by the tree-level superpotential of Eq. (5.1)

$$W(x) := g_0 + \sum_{k=1}^{n+1} \frac{g_k}{k} x^k, \quad g_{n+1} = 1 \ . \tag{5.4}$$

[1]We reserve the letter N for the number of colours in a $U(N)$ gauge theory. It is important to distinguish between N in the gauge theory and $\hat{N}$ in the matrix model.

[2]Here and in the following we will write γ for both the function and its image.

Without loss of generality we have chosen $g_{n+1} = 1$. The only restriction for the other complex parameters $\{g_k\}_{k=0,\ldots n}$, collectively denoted by g, comes from the fact that the n critical points μ_i of W should not be degenerate, i.e. $W''(\mu_i) \neq 0$ if $W'(x) = \prod_{i=1}^n (x - \mu_i)$. Then the partition function of the holomorphic one-matrix model is

$$Z(\Gamma(\gamma), g, g_s, \hat{N}) := C_{\hat{N}} \int_{\Gamma(\gamma)} \mathrm{d}M \ \exp\left(-\frac{1}{g_s}\operatorname{tr} W(M)\right) , \tag{5.5}$$

where g_s is a positive coupling constant and $C_{\hat{N}}$ is some normalisation factor. To avoid cluttering the notation we will omit the dependence on γ and g and write $Z(g_s, \hat{N}) := Z(\Gamma(\gamma), g, g_s, \hat{N})$. As usual [59] one diagonalises M and performs the integral over the diagonalising matrices. The constant $C_{\hat{N}}$ is chosen in such a way that one arrives at

$$Z(g_s, \hat{N}) = \frac{1}{\hat{N}!} \int_\gamma \mathrm{d}\lambda_1 \ldots \int_\gamma \mathrm{d}\lambda_{\hat{N}} \exp\left(-\hat{N}^2 S(g_s, \hat{N}; \lambda_m)\right) =: e^{-F(g_s, \hat{N})} , \tag{5.6}$$

where

$$S(g_s, \hat{N}; \lambda_m) = \frac{1}{\hat{N}^2 g_s} \sum_{m=1}^{\hat{N}} W(\lambda_m) - \frac{1}{\hat{N}^2} \sum_{p \neq q} \ln(\lambda_p - \lambda_q) . \tag{5.7}$$

See [95] for more details.

The convergence of the λ_m integrals depends on the polynomial W and the choice of the path γ. For instance, it is clear that once we take W to be odd, γ cannot coincide with the real axis but has to be chosen differently. For given W the asymptotic part of the complex plane ($|x|$ large) can be divided into convergence domains $G_l^{(c)}$ and divergence domains $G_l^{(d)}$, $l = 1, \ldots n+1$, where $e^{-\frac{1}{g_s}W(x)}$ converges, respectively diverges as $|x| \to \infty$. To see this in more detail take $x = re^{i\theta}$ and $g_k = r_k e^{-i\theta_k}$, with $r, r_k \geq 0$, $\theta, \theta_k \in [0, 2\pi)$ for $k = 1, \ldots n$ and $r_{n+1} = 1$, $\theta_{n+1} = 0$, and plug it into the potential,

$$W(re^{i\theta}) = g_0 + \sum_{k=1}^{n+1} \frac{r_k r^k}{k} \cos(k\theta - \theta_k) + i \sum_{k=1}^{n+1} \frac{r_k r^k}{k} \sin(k\theta - \theta_k) . \tag{5.8}$$

The basic requirement is that $\left|e^{-\frac{1}{g_s}W(re^{i\theta})}\right| = e^{-\frac{1}{g_s}ReW(re^{i\theta})}$ should vanish for $r \to \infty$. If we fix θ, s.t. $\cos((n+1)\theta) \neq 0$, then $e^{-\frac{1}{g_s}W(re^{i\theta})}$ decreases exponentially for $r \to \infty$ if and only if $\cos((n+1)\theta) > 0$ which gives

$$\theta = \frac{\alpha}{n+1} + \pi\frac{2(l-1)}{n+1} \quad \text{with } l = 1, 2 \ldots, n+1 \text{ and } \alpha \in (-\pi/2, \pi/2) . \tag{5.9}$$

This defines $n+1$ open wedges in $\mathbb{C}$ with apex at the origin, which we denote by $G_l^{(c)}$, $l = 1 \ldots, n+1$. The complementary sectors $G_l^{(d)}$ are regions where $\cos((n+1)\theta) < 0$, i.e.

$$\theta = \frac{\alpha}{n+1} + \pi\frac{2l-1}{n+1} \quad \text{with } l = 1, 2 \ldots, n+1 \text{ and } \alpha \in (-\pi/2, \pi/2) . \tag{5.10}$$

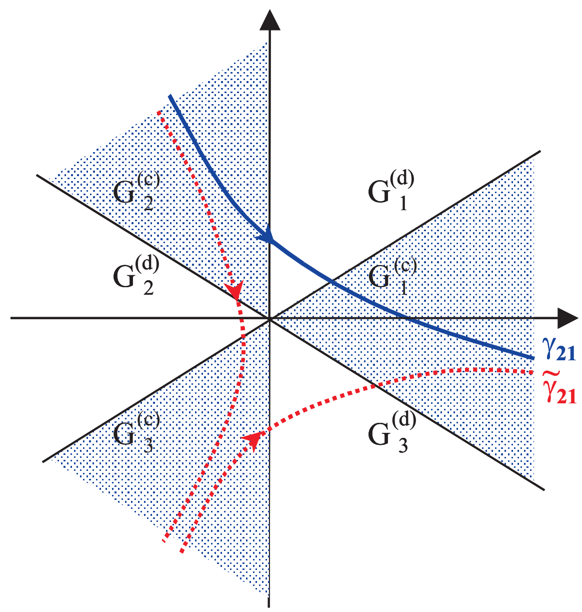

Figure 5.1: Example of convergence and divergence domains for $n = 2$ and a possible choice of γ_{21}. Because of holomorphicity the path can be deformed without changing the partition function, for instance one could use the path $\tilde{\gamma}_{21}$ instead.

The path γ has to be chosen [95] to go from some convergence domain $G_k^{(c)}$ to some other $G_l^{(c)}$, with $k \neq l$; call such a path γ_{kl}, see Fig. 5.1. Note that, for the case of $n+1$ even, the convergence sectors $G_k^{(c)}$ come in pairs, symmetric with respect to the inversion $x \to -x$, so that $G_1^{(c)}$ and $G_{(n+3)/2}^{(c)}$ lie opposite each other and cover the real axis. Then we can choose γ to coincide with the real axis. In this case the holomorphic matrix model reduces to the eigenvalue representation of the Hermitian matrix model. In the case of odd $n+1$, the image of $G_k^{(c)}$ under $x \to -x$ is $G_{k+[(n+1)/2]}^{(d)}$, and the contour cannot chosen to be the real line. The value of the partition function depends only on the pair (k, l) and, because of holomorphicity, is not sensitive to deformations of γ_{kl}. In particular, instead of γ_{kl} we can make the equivalent choice [P5]

$$\tilde{\gamma}_{kl} = \gamma_{p_1 p_2} \cup \gamma_{p_2 p_3} \cup \ldots \cup \gamma_{p_{n-1} p_n} \cup \gamma_{p_n p_{n+1}} \quad \text{with } p_1 = k,\ p_{n+1} = l, \tag{5.11}$$

as shown in Fig. 5.1. Here we split the path into n components, each component running from one convergence sector to another. Again, due to holomorphicity we can choose the decomposition in such a way that every component $\gamma_{p_i p_{i+1}}$ runs through one of the n critical points of W in $\mathbb{C}$, or at least comes close to it. This choice of $\tilde{\gamma}_{kl}$ will turn out to be very useful to understand the saddle point approximation discussed below. Hence, the partition function and the free energy depend on the pair (k, l), g, g_s and $\hat{N}$. Of course, one can always relate the partition function for arbitrary (k, l) to one with $(k', 1)$, $k' = k - l + 1 \bmod n$, and redefined coupling constants $g_1, \ldots g_{n+1}$.

5.1.2 Perturbation theory and fatgraphs

Later we will discuss a method how one can calculate (at least part of) the free energy non-perturbatively. There is, however, also a Feynman diagram technique that can be used to evaluate the partition function. Here we follow the exposition of [58], where more details can be found. Define a Gaussian expectation value,

$$\langle f(M)\rangle_G := \frac{\int \mathrm{d}M\ f(M)\exp\left(-\frac{1}{g_s}\frac{m}{2}\,\mathrm{tr}\,M^2\right)}{\int \mathrm{d}M\ \exp\left(-\frac{1}{g_s}\frac{m}{2}\,\mathrm{tr}\,M^2\right)}\ , \tag{5.12}$$

and let us for simplicity work with a cubic superpotential in this subsection, $W(x) = \frac{m}{2}x^2 + \frac{g}{3}x^3$, with m, g real and positive. (Note that we take $g_{n+1} = g \neq 1$ in this section, since we want to do perturbation theory in g.) We set $Z^G := C_{\hat{N}} \int \mathrm{d}M\ \exp\left(-\frac{1}{g_s}\frac{m}{2}\,\mathrm{tr}\,M^2\right)$ and expand the interaction term,

$$\begin{aligned} Z(g_s,\hat{N}) &= C_{\hat{N}}\int_\Gamma \mathrm{d}M\ \exp\left(-\frac{1}{g_s}\frac{m}{2}\,\mathrm{tr}\,M^2\right)\sum_{V=0}^{\infty}\frac{1}{V!}\left(-\frac{1}{g_s}\frac{g}{3}\,\mathrm{tr}\,M^3\right)^V \\ &= \sum_{V=0}^{\infty}\frac{1}{V!}Z^G\left\langle\left(-\frac{1}{g_s}\frac{g}{3}\,\mathrm{tr}\,M^3\right)^V\right\rangle_G\ . \end{aligned} \tag{5.13}$$

The standard way to calculate $\langle(-\frac{1}{g_s}\frac{g}{3}\,\mathrm{tr}\,M^3)^V\rangle_G$ is, of course, to introduce sources J with

$$\langle\exp(\,\mathrm{tr}\,JM)\rangle_G = \exp\left(\frac{g_s}{2m}\,\mathrm{tr}\,J^2\right)\ , \tag{5.14}$$

such that one obtains the propagator

$$\langle M_{ij}M_{kl}\rangle_G = \frac{\partial}{\partial J_{ji}}\frac{\partial}{\partial J_{lk}}\langle\exp(\,\mathrm{tr}\,JM)\rangle_G\Big|_{J=0} = \frac{g_s}{m}\delta_{il}\delta_{jk}\ . \tag{5.15}$$

More generally one deduces [58] the *matrix Wick theorem*

$$\left\langle\prod_{(i,j)\in I}M_{ij}\right\rangle_G = \sum_{\text{parings }P}\ \prod_{((i,j),(k,l))\in P}\langle M_{ij}M_{kl}\rangle_G\ , \tag{5.16}$$

where I is an index family containing pairs (i,j), and the sum runs over the set of possibilities to group the index pairs (i,j) again into pairs. Of course the propagator relation tells us that most of these pairings give zero. The remaining ones can be captured by Feynman graphs if we establish the diagrammatic rules of Fig. 5.2. For obvious reasons these Feynman graphs are called *fatgraphs*. We are then left with the relation

$$\langle(\,\mathrm{tr}\,M^k)^V\rangle_G = \sum_{\substack{\text{fatgraphs }\Gamma\text{ with}\\ V\text{ k-valent vertices}}}\mathcal{N}_\Gamma\left(\frac{g_s}{m}\right)^{E(\Gamma)}\hat{N}^{F(\Gamma)}\ , \tag{5.17}$$

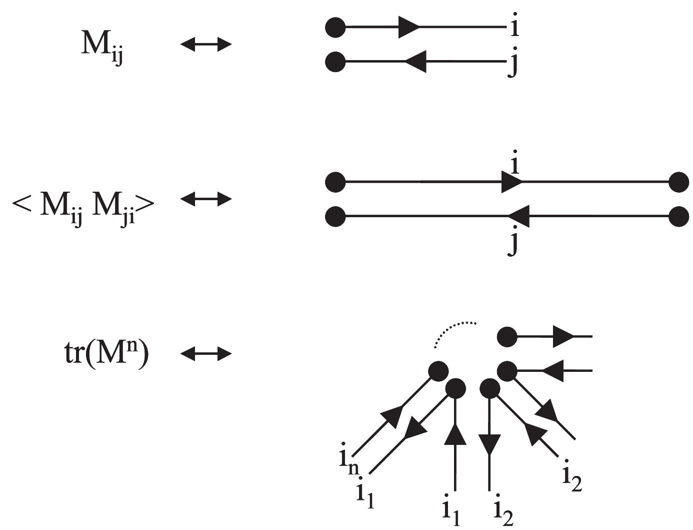

Figure 5.2: The Feynman diagrammatic representation of the matrix M_{ij}, the propagator $\langle M_{ij} M_{ji} \rangle_G$ and an n-valent vertex $\mathrm{tr}\,(M^n)$.

where $F(\Gamma)$ is the number of index loops in the fatgraph Γ, $E(\Gamma)$ is the number of propagators and $\mathcal{N}_\Gamma$ is the number of different ways the propagators can be glued together to build the fatgraph Γ. The partition function for our cubic example can then be expressed as

$$
\begin{aligned}
Z(g_s, \hat{N}) &= \sum_{V=0}^{\infty} \frac{1}{V!} Z^G \left(-\frac{1}{g_s} \frac{g}{3} \right)^V \sum_{\substack{\text{fatgraphs } \Gamma \text{ with} \\ V \text{ 3-valent vertices}}} \mathcal{N}_\Gamma \left(\frac{g_s}{m} \right)^{E(\Gamma)} \hat{N}^{F(\Gamma)} \\
&= Z^G \sum_{\text{fatgraphs } \Gamma} (-g)^{V(\Gamma)} m^{-E(\Gamma)} \frac{1}{|Aut(\Gamma)|} t^F g_s^{2\hat{g}-2} \,, \qquad (5.18)
\end{aligned}
$$

where $|Aut(\Gamma)|$ is the symmetry group of Γ, $\hat{g}$ is the genus of the Riemann surface on which the fatgraph Γ can be drawn (c.f. our discussion in the introduction), and

$$
t := g_s \hat{N} \qquad (5.19)
$$

is the (matrix model) 't Hooft coupling. Similarly, for the free energy we find

$$
F(g_s, t) = \sum_{\text{connected fatgraphs } \Gamma} - (-g)^{V(\Gamma)} m^{-E(\Gamma)} \frac{1}{|Aut(\Gamma)|} t^F g_s^{2\hat{g}-2} - \log(Z^G) \,. \qquad (5.20)
$$

Note that this has precisely the structure

$$
F(g_s, t) = \sum_{\hat{g}=0}^{\infty} \sum_{h=1}^{\infty} F_{\hat{g},h} t^h g_s^{2\hat{g}-2} + \text{non-perturbative} \,, \qquad (5.21)
$$

that we encountered already in the introduction. This tells us that we can not determine the entire free energy from a Feynman diagram expansion, since we also have to take care of the non-perturbative piece. Furthermore, note that in the particular limit in which $\hat{N} \to \infty$ with fixed t only planar diagrams, i.e. those with $\hat{g} = 0$, contribute to the perturbative expansion.

The free energy can then be calculated perturbatively by following a set of Feynman rules:
• To calculate F up to order g^k draw all possible fatgraph diagrams that contain up to k vertices.
• Assign a factor $-\frac{g}{g_s}$ to every vertex.
• Assign a factor $\frac{g_s}{m}$ to every fatgraph propagator.
• Assign a factor $\hat{N}$ to each closed index line.
• Multiply the contribution of a given diagram by $|Aut(\Gamma)|^{-1}$, where $Aut(\Gamma)$ is the automorphism group of the diagram.
• Sum all the contributions and multiply the result by an overall minus sign, which comes from the fact, that $F(g_s, t) = -\log Z(g_s, t)$.

As an example let us calculate the planar free energy up to order g^4. Clearly, we cannot build a vacuum diagram from an odd number of vertices, which is, of course, consistent with $\langle (\operatorname{tr} M^3)^V \rangle_G = 0$ for odd V. The first non-trivial contribution to the free energy comes from $\langle (\operatorname{tr} M^3)^2 \rangle_G$, which is given by the sum of all possibilities to connect two trivalent vertices, see Fig. 5.3. The three diagrams that contribute are

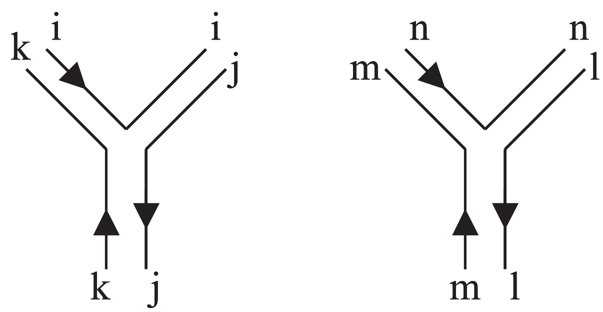

Figure 5.3: To calculate $\langle (\operatorname{tr} M^3)^2 \rangle$ on has to sum over all the possibilities to connect two trivalent vertices.

sketched in Fig. 5.4. According to our rules the first diagram gives $\frac{1}{g_s^2} \frac{t^3 g^2}{m^3} \frac{1}{2}$, the second

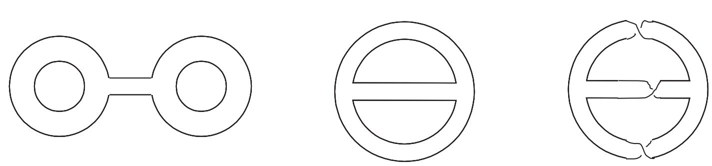

Figure 5.4: From two three-valent vertices one can draw three Feynman diagrams, two of which can be drawn on the sphere, whereas the last one is “nonplanar”.

one contributes $\frac{1}{g_s^2}\frac{t^3 g^2}{m^3}\frac{1}{6}$ and the last one $\frac{tg^2}{m^3}\frac{1}{6}$. Then we have[3]

$$F(g_s,t) \;=\; -\frac{1}{g_s^2}\frac{2}{3}\frac{g^2}{m^3}t^3 - \frac{1}{6}\frac{g^2}{m^3}t + \ldots - \log\left(Z^G\right) \;. \tag{5.22}$$

The terms of order g^4 in $F_0(t)$ can be obtained by writing down all fatgraphs with four vertices that can be drawn on a sphere. They are sketched, together with their symmetry factor $|Aut(\Gamma)|$, in Fig. 5.5. Adding all these contributions to the result of

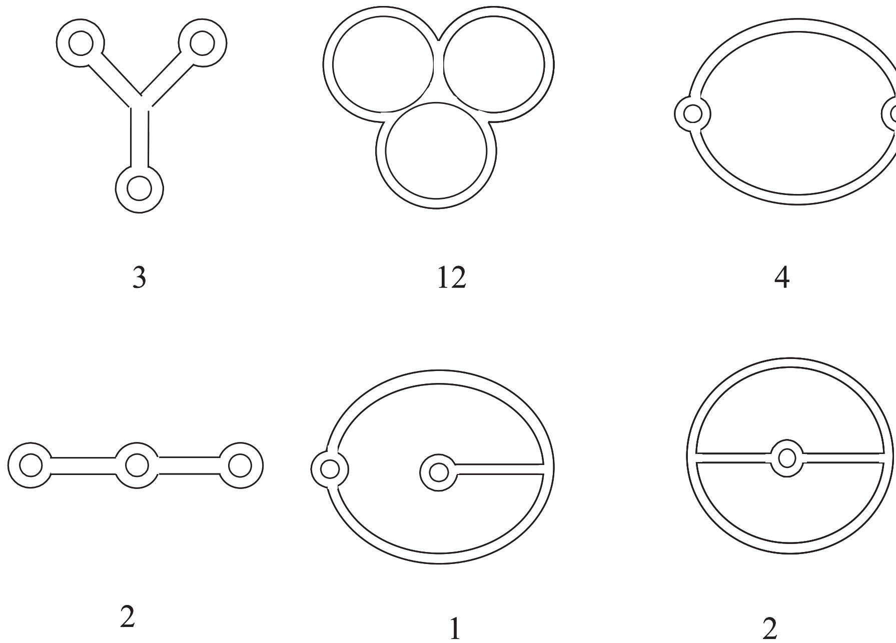

Figure 5.5: All possible planar diagrams containing four vertices, together with their symmetry factors $|Aut(\Gamma)|$.

order g^2 gives an expression for the planar free energy up to order g^4:

$$F_0(t) = -\frac{2}{3}\frac{g^2t^3}{m^3} - \frac{8}{3}\frac{g^4t^4}{m^6} + \ldots + F_0^{np}(t) \;. \tag{5.23}$$

[3]Alternatively we can use the explicit formula for F. From (5.17) one obtains

$$\begin{aligned} F(g_s,t) &= -\log\left(\sum_{V=0}^{\infty}\frac{1}{V!}\left(-\frac{1}{g_s}\frac{g}{3}\right)^V\left\langle\left(\mathrm{tr}\,M^3\right)^V\right\rangle_G\right) - \log\left(Z^G\right) \\ &= -\log\left(1+\frac{1}{2}\frac{1}{g_s^2}\frac{g^2}{3^2}(9+3)N^3\left(\frac{g_s}{m}\right)^3 + \frac{1}{2}\frac{1}{g_s^2}\frac{g^2}{3^2}3N\left(\frac{g_s}{m}\right)^3 + \ldots\right) - \log\left(Z^G\right) \end{aligned}$$

In the second line we used the fact that when writing down all possible pairings one obtains nine times the first diagram of Fig. 5.4, three times the second and three times the last one. Of course, the result coincides with the one obtained using the Feynman rules.

5.1.3 Matrix model technology

Instead of doing perturbation theory one can evaluate the matrix model partition function using some specific matrix model technology. Next, we will recall some of this standard technology adapted to the holomorphic matrix model. Let us first assume that the path γ consists of a single connected piece. The case (5.11) will be discussed later on. Let s be the length coordinate of the path γ, centered at some point on γ, and let $\lambda(s)$ denote the parameterisation of γ with respect to this coordinate. Then, for an eigenvalue λ_m on γ, one has $\lambda_m = \lambda(s_m)$ and the partition function (5.6) can be rewritten as

$$Z(g_s, \hat{N}) = \frac{1}{\hat{N}!} \int_{\mathbb{R}} \mathrm{d}s_1 \ldots \int_{\mathbb{R}} \mathrm{d}s_{\hat{N}} \prod_{l=1}^{\hat{N}} \dot{\lambda}(s_l) \exp\left(-\hat{N}^2 S(g_s, \hat{N}; \lambda(s_m))\right) \ . \tag{5.24}$$

The spectral density is defined as

$$\rho(s, s_m) := \frac{1}{\hat{N}} \sum_{m=1}^{\hat{N}} \delta(s - s_m) \ , \tag{5.25}$$

so that ρ is normalised to one, $\int_{-\infty}^{\infty} \rho(s, s_m)\mathrm{d}s = 1$. The normalised trace of the resolvent of the matrix M is given by

$$\omega(x, s_m) := \frac{1}{\hat{N}} \operatorname{tr} \frac{1}{x - M} = \frac{1}{\hat{N}} \sum_{m=1}^{\hat{N}} \frac{1}{x - \lambda(s_m)} = \int \mathrm{d}s \frac{\rho(s, s_m)}{x - \lambda(s)} \ , \tag{5.26}$$

for $x \in \mathbb{C}$. Following [95] we decompose the complex plane into domains D_i, $i \in \{1, \ldots, n\}$, with mutually disjoint interior, $(\cup_i \overline{D}_i = \mathbb{C}, \ \ D_i \cap D_j = \emptyset \ \text{ for } i \neq j)$. These domains are chosen in such a way that γ intersects each $\overline{D}_i$ along a single line segment Δ_i, and $\cup_i \Delta_i = \gamma$. Furthermore, μ_i, the i-th critical point of W, should lie in the interior of D_i. One defines

$$\chi_i(M) := \int_{\partial D_i} \frac{\mathrm{d}x}{2\pi i} \frac{1}{x - M} \ , \tag{5.27}$$

(which projects on the space spanned by the eigenvectors of M whose eigenvalues lie in D_i), and the filling fractions $\tilde{\sigma}_i(\lambda_m) := \frac{1}{\hat{N}} \operatorname{tr} \chi_i(M)$ and

$$\sigma_i(s_m) := \tilde{\sigma}_i(\lambda(s_m)) = \int \mathrm{d}s \ \rho(s, s_m) \chi_i(\lambda(s)) = \int_{\partial D_i} \frac{\mathrm{d}x}{2\pi i} \ \omega(x, s_m) \ , \tag{5.28}$$

(which count the eigenvalues in the domain D_i, times $1/\hat{N}$). Obviously

$$\sum_{i=1}^{n} \sigma_i(s_m) = 1 \ . \tag{5.29}$$

Loop equations

Next we apply the methods of [94] to derive the loop equations of the holomorphic matrix model. We define the expectation value

$$\langle h(\lambda_m)\rangle := \frac{1}{Z(g_s,\hat{N})}\cdot\frac{1}{\hat{N}!}\int_\gamma \mathrm{d}\lambda_1\ldots\int_\gamma \mathrm{d}\lambda_{\hat{N}}\; h(\lambda_m)\exp\left(-\hat{N}^2 S(g_s,\hat{N};\lambda_m)\right)\;. \tag{5.30}$$

From the translational invariance of the measure one finds the identity [94], [95]

$$\int_\gamma \mathrm{d}\lambda_1\ldots\int_\gamma \mathrm{d}\lambda_{\hat{N}}\sum_{m=1}^{\hat{N}}\frac{\partial}{\partial\lambda_m}\left[\prod_{k\neq l}(\lambda_k-\lambda_l)e^{-\frac{1}{g_s}\sum_{j=1}^{\hat{N}}W(\lambda_j)}\frac{1}{x-\lambda_m}\right]=0\;. \tag{5.31}$$

Evaluating the derivative gives

$$\left\langle\sum_{m=1}^{\hat{N}}\frac{1}{(x-\lambda_m)^2}-\frac{1}{g_s}\sum_{m=1}^{\hat{N}}\frac{W'(\lambda_m)}{x-\lambda_m}+2\sum_{m=1}^{\hat{N}}\sum_{\substack{l=1\\ l\neq m}}^{\hat{N}}\frac{1}{(\lambda_m-\lambda_l)(x-\lambda_m)}\right\rangle=0\;. \tag{5.32}$$

Using

$$\frac{1}{(x-\alpha)(x-\beta)}=\frac{1}{\alpha-\beta}\left[\frac{1}{x-\alpha}-\frac{1}{x-\beta}\right] \tag{5.33}$$

we find

$$\sum_{m=1}^{\hat{N}}\frac{1}{(x-\lambda_m)^2}+2\sum_{m=1}^{\hat{N}}\sum_{\substack{l=1\\ l\neq m}}^{\hat{N}}\frac{1}{(\lambda_m-\lambda_l)(x-\lambda_m)}=\sum_{l,m=1}^{\hat{N}}\frac{1}{x-\lambda_m}\frac{1}{x-\lambda_l} \tag{5.34}$$

and therefore

$$\left\langle\omega(x;s_m)^2-\frac{1}{t\hat{N}}\sum_{m=1}^{\hat{N}}\frac{W'(\lambda(s_m))}{x-\lambda(s_m)}\right\rangle=0\;. \tag{5.35}$$

If we define the polynomial

$$f(x;s_m):=-\frac{4t}{\hat{N}}\sum_{m=1}^{\hat{N}}\frac{W'(x)-W'(\lambda(s_m))}{x-\lambda(s_m)}=-4t\int\mathrm{d}s\;\rho(s;s_m)\frac{W'(x)-W'(\lambda(s))}{x-\lambda(s)}\;, \tag{5.36}$$

we obtain the *loop equations*

$$\langle\omega(x;s_m)^2\rangle-\frac{1}{t}W'(x)\langle\omega(x;s_m)\rangle-\frac{1}{4t^2}\langle f(x;s_m)\rangle=0\;. \tag{5.37}$$

Equations of motion

It will be useful to define an effective action as

$$\begin{aligned} S_{eff}(g_s,\hat{N};s_m) &:= S(g_s,\hat{N};\lambda(s_m))-\frac{1}{\hat{N}^2}\sum_{m=1}^{\hat{N}}\ln(\dot{\lambda}(s_m)) \\ &= \int\mathrm{d}s\;\rho(s;s_m)\left(\frac{1}{t}W(\lambda(s))-\frac{1}{\hat{N}}\ln(\dot{\lambda}(s))-\mathcal{P}\int\mathrm{d}s'\;\rho(s';s_p)\ln(\lambda(s)-\lambda(s'))\right) \end{aligned} \tag{5.38}$$

so that

$$Z(g_s, \hat{N}) = \frac{1}{\hat{N}!} \int \mathrm{d}s_1 \ldots \int \mathrm{d}s_{\hat{N}} \; \exp\left(-\hat{N}^2 S_{eff}(g_s, \hat{N}; s_m)\right) \; . \tag{5.39}$$

Note that the principal value is defined as

$$\mathcal{P} \; \ln\left(\lambda(s) - \lambda(s')\right) = \frac{1}{2} \lim_{\epsilon \to 0} \left[\ln\left(\lambda(s) - \lambda(s') + i\epsilon\dot{\lambda}(s)\right) + \ln\left(\lambda(s) - \lambda(s') - i\epsilon\dot{\lambda}(s)\right)\right] \; . \tag{5.40}$$

The equations of motion corresponding to this effective action, $\frac{\delta S_{eff}}{\delta s_m} = 0$, read

$$\frac{1}{t} W'(\lambda(s_m)) = \frac{2}{\hat{N}} \sum_{p=1,\; p \neq m}^{\hat{N}} \frac{1}{\lambda(s_m) - \lambda(s_p)} + \frac{1}{\hat{N}} \frac{\ddot{\lambda}(s_m)}{\dot{\lambda}(s_m)^2} \; . \tag{5.41}$$

Using these equations of motion one can show that

$$\begin{aligned} &\omega(x, s_m)^2 - \frac{1}{t} W'(x)\omega(x, s_m) - \frac{1}{4t^2} f(x, s_m) + \\ &+ \frac{1}{\hat{N}} \frac{\mathrm{d}}{\mathrm{d}x} \omega(x, s_m) + \frac{1}{\hat{N}^2} \sum_{m=1}^{\hat{N}} \frac{\ddot{\lambda}(s_m)}{\dot{\lambda}(s_m)^2} \frac{1}{x - \lambda(s_m)} = 0 \; . \end{aligned} \tag{5.42}$$

Solutions of the equations of motion

Note that in general the effective action is a complex function of the real s_m. Hence, in general, i.e. for a generic path γ_{kl} with parameterisation $\lambda(s)$, there will be no solution to (5.41). One clearly expects that the existence of solutions must constrain the path $\lambda(s)$ appropriately. Let us study this in more detail.

Recall that we defined the domains D_i in such a way that $\mu_i \subset D_i$. Let $\hat{N}_i$ be the number of eigenvalues $\lambda(s_m)$ which lie in the domain D_i, so that $\sum_{i=1}^n \hat{N}_i = \hat{N}$, and denote them by $\lambda(s_a^{(i)})$, $a \in \{1, \ldots \hat{N}_i\}$.

Solving the equations of motion in general is a formidable problem. To get a good idea, however, recall the picture of $\hat{N}_i$ fermions filled into the i-th "minimum" of $\frac{1}{t}W$ [90]. For small t the potential is deep and the fermions are located not too far from the minimum, in other words all the eigenvalues are close to μ_i. To be more precise consider (5.41) and drop the last term, an approximation that will be justified momentarily. Let us take t to be small and look for solutions[4] $\lambda(s_a^{(i)}) = \mu_i + \sqrt{t}\delta\lambda_a^{(i)}$, where $\delta\lambda_a^{(i)}$ is of order one. So, we assume that the eigenvalues $\lambda(s_a^{(i)})$ are not too far from the critical point μ_i. Then the equation reads

$$W''(\mu_i)\delta\lambda_a^{(i)} = \frac{2}{\hat{N}} \sum_{b=1,\; b \neq a}^{\hat{N}_i} \frac{1}{\delta\lambda_a^{(i)} - \delta\lambda_b^{(i)}} + o(\sqrt{t}) \; , \tag{5.43}$$

[4] One might try the general ansatz $\lambda(s_a^{(i)}) = \mu_i + \epsilon\delta\lambda_a^{(i)}$ but it turns out that a solution can be found only if $\epsilon \sim \sqrt{t}$.

so we effectively reduced the problem to finding the solution for n distinct quadratic potentials. If we set $z_a := \sqrt{\frac{\hat{N}W''(\mu_i)}{2}}\delta\lambda_a^{(i)}$ and neglect the $o(\sqrt{t})$-terms this gives

$$z_a = \sum_{b=1,\ b\neq a}^{\hat{N}_i} \frac{1}{z_a - z_b} \ , \tag{5.44}$$

which can be solved explicitly for small $\hat{N}_i$. It is obvious that $\sum_{a=1}^{\hat{N}_i} z_a = 0$, and one finds that there is a unique solution (up to permutations) with the z_a symmetrically distributed around 0 on the real axis. This justifies a posteriori that we really can neglect the term proportional to the second derivative of $\lambda(s)$, at least to leading order. Furthermore, setting $W''(\mu_i) = |W''(\mu_i)|e^{i\phi_i}$ one finds that the $\lambda(s_a^i)$ sit on a tilted line segment around μ_i where the angle of the tilt is given by $-\phi_i/2$. This means for example that for a potential with $W'(x) = x(x-1)(x+1)$ the eigenvalues are distributed on the real axis around ± 1 and on the imaginary axis around 0. Note further that, in general, the reality of z_a implies that $\frac{W''(\mu_i)}{2}\left(\delta\lambda_a^{(i)}\right)^2 > 0$ which tells us that, close to μ_i, $W(\lambda(s)) - W(\mu_i)$ is real with a *minimum* at $\lambda(s) = \mu_i$.

So we have found that the path γ_{kl} has to go through the critical points μ_i with a tangent direction fixed by the phase of the second derivative of W. On the other hand, we know that the partition function does not depend on the form of the path γ_{kl}. Of course, there is no contradiction: if one wants to compute the partition function from a saddle point expansion, as we will do below, and as is implicit in the planar limit, one has to make sure that one expands around solutions of (5.41) and the existence of these solutions imposes conditions on how to choose the path γ_{kl}. From now on we will assume that the path is chosen in such a way that it satisfies all these constraints. Furthermore, for later purposes it will be useful to use the path $\tilde{\gamma}_{kl}$ of (5.11) chosen such that its part $\gamma_{p_i p_{i+1}}$ goes through all $\hat{N}_i$ solutions $\lambda_a^{(i)}$, $a = 1, \ldots \hat{N}_i$, and lies entirely in D_i, see Fig. 5.6.

It is natural to assume that these properties together with the uniqueness of the solution (up to permutations) extend to higher numbers of $\hat{N}_i$ as well. Of course once one goes beyond the leading order in $\sqrt{t}$ the eigenvalues are no longer distributed on a straight line, but on a line segment that is bent in general and that might or might not pass through μ_i.

The large $\hat{N}$ limit

We are interested in the large $\hat{N}$ limit of the matrix model free energy. It is well known that the expectation values of the relevant quantities like ρ or ω have expansions of the form

$$\langle \rho(s, s_m) \rangle = \sum_{I=0}^{\infty} \rho_I(s) \hat{N}^{-I} \quad , \quad \langle \omega(x, s_m) \rangle = \sum_{I=0}^{\infty} \omega_I(x) \hat{N}^{-I} \ . \tag{5.45}$$

Clearly, $\omega_0(x)$ is related to $\rho_0(s)$ by the large $\hat{N}$ limit of (5.26), namely

$$\omega_0(x) = \int \mathrm{d}s \frac{\rho_0(s)}{x - \lambda(s)} \ . \tag{5.46}$$

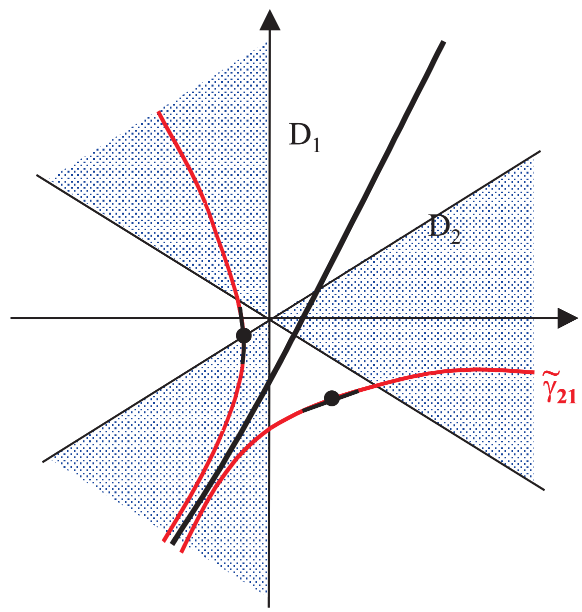

Figure 5.6: For the cubic potential of Fig. 5.1 we show the choice of the domains D_1 and D_2 (to the left and right of the bold line) and of the path $\tilde{\gamma}_{21}$ with respect to the two critical points, as well as the cuts that form around these points.

We saw already that an eigenvalue ensemble that solves the equations of motion is distributed along line segments around the critical points μ_i. In the limit $\hat{N} \to \infty$ this will turn into a continuous distribution on the segments $\mathcal{C}_i$, through or close to the critical points of W. Then $\rho_0(s)$ has support only on these $\mathcal{C}_i$ and $\omega_0(x)$ is analytic in $\mathbb{C}$ with cuts $\mathcal{C}_i$. Conversely, $\rho_0(s)$ is given by the discontinuity of $\omega_0(x)$ across its cuts:

$$\rho_0(s) := \dot{\lambda}(s) \lim_{\epsilon \to 0} \frac{1}{2\pi i} [\omega_0(\lambda(s) - i\epsilon\dot{\lambda}(s)) - \omega_0(\lambda(s) + i\epsilon\dot{\lambda}(s))] \ . \tag{5.47}$$

The planar limit we are interested in is $\hat{N} \to \infty$, $g_s \to 0$ with $t = g_s \hat{N}$ held fixed. Hence we rewrite all $\hat{N}$ dependence as a g_s dependence and consider the limit $g_s \to 0$. Then, the equation of motion (5.41) reduces to

$$\frac{1}{t} W'(\lambda(s)) = 2\mathcal{P} \int \mathrm{d}s' \ \frac{\rho_0(s')}{\lambda(s) - \lambda(s')} \ . \tag{5.48}$$

Note that this equation is only valid for those s where eigenvalues exist, i.e. where $\rho_0(s) \neq 0$. In principle one can use this equation to compute the planar eigenvalue distribution $\rho_0(s)$ for given W'.

The leading terms in the expansions (5.45) for $\langle \rho(s, s_m) \rangle$ or $\langle \omega(x, s_m) \rangle$, i.e. $\rho_0(s)$ or $\omega_0(x)$, can be calculated from a saddle point approximation, where the $\{s_m\}$ are given by a solution $\{s_m^*\}$ of (5.41): $\rho_0(s) = \rho(s; s_m^*)$, or explicitly from eq. (5.25)

$$\rho_0(s) = \frac{1}{\hat{N}} \sum_{m=1}^{\hat{N}} \delta(s - s_m^*) \ . \tag{5.49}$$

Note in particular that $\rho_0(s)$ is manifestly real. This is by no means obvious for the full expectation value of $\rho(s, s_m)$ since it must be computed by averaging with respect to a complex measure. In the planar limit, however, the quantum integral is essentially localized at a single classical configuration $\{s_m^*\}$ and this is why $\rho_0(s)$ is real. The large $\hat{N}$ limit of the resolvent $\omega_0(x) = \omega(x; s_m^*)$ then is still given by (5.46).
This prescription to compute expectation values of operators in the large $\hat{N}$ limit is true for all "microscopic" operators, i.e. operators that do not modify the saddle point equations (5.41). (Things would be different for "macroscopic"operators like $e^{\hat{N}\sum_{p=1}^{\hat{N}} V(\lambda_p)}$.) In particular, this shows that expectation values factorise in the large $\hat{N}$ limit.

Riemann surfaces and planar solutions
This factorisation of expectation values shows that in the large $\hat{N}$ limit the loop equation (5.37) reduces to the algebraic equation

$$\omega_0(x)^2 - \frac{1}{t}W'(x)\omega_0(x) - \frac{1}{4t^2}f_0(x) = 0 \; , \tag{5.50}$$

where

$$f_0(x) = -4t \int \mathrm{d}s \; \rho_0(s) \frac{W'(x) - W'(\lambda(s))}{x - \lambda(s)} \tag{5.51}$$

is a polynomial of degree $n-1$ with leading coefficient $-4t$. Note that this coincides with the planar limit of equation (5.42). If we define

$$y_0(x) := W'(x) - 2t\omega_0(x) \; , \tag{5.52}$$

then y_0 is one of the branches of the algebraic curve

$$y^2 = W'(x)^2 + f_0(x) \; , \tag{5.53}$$

as can be seen from (5.50). This is in fact an extremely interesting result. Not only is it quite amazing that a Riemann surface arises in the large $\hat{N}$ limit of the holomorphic matrix model, but it is actually the Riemann surface (4.76), which we encountered when we were calculating the integrals of Ω over three-cycles in X_{def}. This is important because we can now readdress the problem of calculating integrals on this Riemann surface. In fact, the matrix model will provide us with the techniques that are necessary to solve this problem.

On the curve (5.53) we use the same conventions as in section 4.2.3, i.e. $y_0(x)$ is defined on the upper sheet and cycles and orientations are chosen as in Fig. 4.2 and Fig. 4.1.

Solving a matrix model in the planar limit means to find a normalised, real, non-negative $\rho_0(s)$ and a path $\tilde{\gamma}_{kl}$ which satisfy (5.46), (5.48) and (5.50/5.51) for a given potential $W(z)$ and a given asymptotics (k, l) of γ.

Interestingly, for any algebraic curve (5.53) there is a contour $\tilde{\gamma}_{kl}$ supporting a *formal* solution of the matrix model in the planar limit. To construct it start from an

arbitrary polynomial $f_0(x)$ or order $n-1$, with leading coefficient $-4t$, which is given together with the potential $W(x)$ of order $n+1$. The corresponding Riemann surface is given by (5.53), and we denote its branch points by $a_i^\pm$ and choose branch-cuts $\mathcal{C}_i$ between them. We can read off the two solutions y_0 and $y_1 = -y_0$ from (5.53), where we take y_0 to be the one with a behaviour $y_0 \stackrel{x\to\infty}{\to} +W'(x)$. $\omega_0(x)$ is defined as in (5.52) and we choose a path $\tilde{\gamma}_{kl}$ such that $\mathcal{C}_i \subset \tilde{\gamma}_{kl}$ for all i. Then the formal planar spectral density satisfying all the requirements can be determined from (5.47) (see [95]). However, in general, this will lead to a *complex* distribution $\rho_0(s)$. This can be understood from the fact the we constructed $\rho_0(s)$ from a completely arbitrary hyperelliptic Riemann surface. However, in the matrix model the algebraic curve (5.53) is *not* general, but the coefficients α_k of $f_0(x)$ are constraint. This can be seen by computing the filling fractions $\langle\sigma_i(s_m)\rangle$ in the planar limit where they reduce to $\sigma_i(s_m^*)$. They are given by

$$\nu_i^* := \sigma_i(s_m^*) = \frac{1}{2\pi i}\int_{\partial D_i}\omega_0(x)\mathrm{d}x = \frac{1}{4\pi i t}\int_{A^i} y_0(x)\mathrm{d}x = \int_{\gamma^{-1}(\mathcal{C}_i)}\rho_0(s)\mathrm{d}s \ , \qquad (5.54)$$

which must be real and non-negative. Here we used the fact that the D_i were chosen such that $\gamma_{p_ip_{i+1}} \subset D_i$ and therefore $\mathcal{C}_i \subset D_i$, so for D_i on the upper plane, ∂D_i is homotopic to $-A^i$. Hence, $\mathrm{Im}\left(i\int_{A^i} y(x)\mathrm{d}x\right) = 0$ which constrains the α_k. We conclude that to construct distributions $\rho_0(s)$ that are relevant for the matrix model one can proceed along the lines described above, but one has to impose the additional constraint that $\rho_0(s)$ is *real* [P5]. As for finite $\hat{N}$, this will impose conditions on the possible paths $\tilde{\gamma}_{kl}$ supporting the eigenvalue distributions.

To see this, we assume that the coefficients α_k in $f_0(x)$ are small, so that the lengths of the cuts are small compared to the distances between the different critical points: $|a_i^+ - a_i^-| \ll |\mu_i - \mu_j|$. Then in first approximation the cuts are straight line segments between a_i^+ and a_i^-. For x close to the cut $\mathcal{C}_i$ we have $y^2 \approx (x-a_i^+)(x-a_i^-)\prod_{j\neq i}(\mu_i - \mu_j)^2 = (x-a_i^+)(x-a_i^-)(W''(\mu_i))^2$. If we set $W''(\mu_i) = |W''(\mu_i)|e^{i\phi_i}$ and $a_i^+ - a_i^- = r_i e^{i\psi_i}$, then, on the cut $\mathcal{C}_i$, the path γ is parameterised by $\lambda(s) = \frac{a_i^+ - a_i^-}{|a_i^+ - a_i^-|}s = se^{i\psi_i}$, and we find from (5.47)

$$\begin{aligned}\rho_0(s) &= \frac{1}{2\pi t}\sqrt{|\lambda(s)-a_1^+|}\sqrt{|\lambda(s)-a_1^-|}\ |W''(\mu_i)|e^{i(\phi_i+2\psi_i)} \\ &= \frac{1}{2\pi t}\sqrt{|\lambda(s)-a_1^+|}\sqrt{|\lambda(s)-a_1^-|}\ W''(\mu_i)(\dot{\lambda}(s))^2 \ . \end{aligned} \qquad (5.55)$$

So reality and positivity of $\rho_0(s)$ lead to conditions on the orientation of the cuts in the complex plane, i.e. on the path γ:

$$\psi_i = -\phi_i/2 \qquad , \qquad W''(\mu_i)(\dot{\lambda}(s))^2 > 0 \ . \qquad (5.56)$$

These are precisely the conditions we already derived for the case of finite $\hat{N}$. We see that the two approaches are consistent and, for given W and fixed $\hat{N}_i$ respectively ν_i^*, lead to a unique[5] solution $\{\lambda(s), \rho_0(s)\}$ with real and positive eigenvalue distribution.

[5] To be more precise the path $\tilde{\gamma}_{kl}$ is not entirely fixed. Rather, for every piece $\tilde{\gamma}_{p_ip_{i+1}}$ we have the requirement that $\mathcal{C}_i \subset \tilde{\gamma}_{p_ip_{i+1}}$.

Note again that the requirement of reality and positivity of $\rho_0(s)$ constrains the phases of $a_i^+ - a_i^-$ and hence the coefficients α_k of $f_0(x)$.

5.1.4 The saddle point approximation for the partition function

Recall that our goal is to calculate the integrals of $\zeta = y\mathrm{d}x$ on the Riemann surface (5.53) using matrix model techniques. The tack will be to establish relations for these integrals which are similar to the special geometry relations (5.2). After we have obtained a clear cut understanding of the function $\mathcal{F}$ appearing in these relations, we can use the matrix model to calculate $\mathcal{F}$ and therefore the integrals. A natural candidate for this function $\mathcal{F}$ is the free energy of the matrix model, or rather, since we are working in the large $\hat{N}$ limit, its planar component $F_0(t)$. However, $F_0(t)$ depends on t only and therefore it cannot appear in relations like (5.2). To remedy this we introduce a set of sources J_i and obtain a free energy that depends on more variables. In this subsection we evaluate this source dependent free energy and its Legendre transform $\mathcal{F}_0(t,S)$ in the planar limit using a saddle point expansion [P5].

We start by coupling sources to the filling fractions,[6]

$$\begin{aligned} Z(g_s,\hat{N},J) &:= \frac{1}{\hat{N}!}\int_\gamma \mathrm{d}\lambda_1 \ldots \int_\gamma \mathrm{d}\lambda_{\hat{N}} \exp\left(-\hat{N}^2 S(g_s,\hat{N};\lambda_m) - \frac{\hat{N}}{g_s}\sum_{i=1}^{n-1} J_i\tilde{\sigma}_i(\lambda_m)\right) \\ &= \exp\Big(-F(g_s,\hat{N},J)\Big) \ . \end{aligned} \tag{5.57}$$

where $J := \{J_1,\ldots,J_{n-1}\}$. Note that because of the constraint $\sum_{i=1}^n \tilde{\sigma}_i(\lambda_m) = 1$, $\tilde{\sigma}_n(\lambda_m)$ is not an independent quantity and we can have only $n-1$ sources. This differs from the treatment in [95] and has important consequences, as we will see below. We want to evaluate this partition function for $\hat{N}\to\infty$, $t = g_s\hat{N}$ fixed, from a saddle point approximation. We therefore use the path $\tilde{\gamma}_{kl}$ from (5.11) that was chosen in such a way that the equation of motion (5.41) has solutions s_m^* and, for large $\hat{N}$, $\mathcal{C}_i \subset \gamma_{p_ip_{i+1}}$. It is only then that the saddle point expansion converges and makes sense. Obviously then each integral $\int_\gamma \mathrm{d}\lambda_m$ splits into a sum $\sum_{i=1}^n \int_{\gamma_{p_ip_{i+1}}} \mathrm{d}\lambda_m$. Let $s^{(i)} \in \mathbb{R}$ be the length coordinate on $\gamma_{p_ip_{i+1}}$, so that $s^{(i)}$ runs over all of $\mathbb{R}$. Furthermore, $\tilde{\sigma}_i(\lambda_m)$ only depends on the number $\hat{N}_i$ of eigenvalues in $\tilde{\gamma}_{kl}\cap D_i = \gamma_{p_ip_{i+1}}$. Then the partition function (5.57) is a sum of contributions with fixed $\hat{N}_i$ and we rewrite is as

$$Z(g_s,N,J) = \sum_{\substack{\sum_i \hat{N}_i = \hat{N} \\ \hat{N}_1,\ldots,\hat{N}_n}} Z(g_s,\hat{N},\hat{N}_i) e^{-\frac{1}{g_s}\sum_{i=1}^{n-1} J_i\hat{N}_i} \ , \tag{5.58}$$

[6] Note that $\exp\left(-\frac{\hat{N}^2}{t}\sum_{i=1}^{n-1} J_i\sigma_i(s_m)\right)$ looks like a macroscopic operator that changes the equations of motion. However, because of the special properties of $\sigma_i(s_m)$ we have $\frac{\partial}{\partial s_n}\sigma_i(s_m) = \frac{1}{\hat{N}}\int_{\partial D_i}\frac{\mathrm{d}x}{2\pi i}\frac{\dot{\lambda}(s_n)}{(x-\lambda(s_n))^2}$. In particular, for the path $\tilde{\gamma}_{kl}$ that will be chosen momentarily and the corresponding domains D_i the eigenvalues λ_m cannot lie on ∂D_i. Hence, $\frac{\partial}{\partial s_n}\sigma_i(s_m) = 0$ and the equations of motion remain unchanged.

where now

$$
\begin{aligned}
&Z(g_s, \hat{N}, \hat{N}_i) = \\
&= \frac{1}{\hat{N}_1 \ldots \hat{N}_n!} \int_{\gamma_{p_1 p_2}} \mathrm{d}\lambda_1^{(1)} \ldots \int_{\gamma_{p_1 p_2}} \mathrm{d}\lambda_{\hat{N}_1}^{(1)} \ldots \int_{\gamma_{p_n p_{n+1}}} \mathrm{d}\lambda_1^{(n)} \ldots \int_{\gamma_{p_n p_{n+1}}} \mathrm{d}\lambda_{\hat{N}_n}^{(n)} \times \\
&\times \exp\left(-\frac{t^2}{g_s^2} S(g_s, t; \lambda_k^{(i)}) \right) \\
&=: \exp\left(\tilde{\mathcal{F}}(g_s, t, \hat{N}_i) \right)
\end{aligned}
\tag{5.59}
$$

is the partition function with the additional constraint that precisely $\hat{N}_i$ eigenvalues lie on $\gamma_{p_i p_{i+1}}$. Note that it depends on $g_s, t = g_s \hat{N}$ and $\hat{N}_1, \ldots \hat{N}_{n-1}$ only, as $\sum_{i=1}^n \hat{N}_i = \hat{N}$. Now that these numbers have been fixed, there is precisely one solution to the equations of motion, i.e. a *unique* saddle-point configuration, up to permutations of the eigenvalues, on *each* $\gamma_{p_i p_{i+1}}$. These permutations just generate a factor $\prod_i \hat{N}_i!$ which cancels the corresponding factor in front of the integral. As discussed above, it is important that we have chosen the $\gamma_{p_i p_{i+1}}$ to support this saddle point configuration close to the critical point μ_i of W. Moreover, since $\gamma_{p_i p_{i+1}}$ runs from one convergence sector to another and by (5.56) the saddle point really is dominant (stable), the "one-loop" and other higher order contributions are indeed subleading as $g_s \to 0$ with $t = g_s \hat{N}$ fixed. This is why we had to be so careful about the choice of our path γ as being composed of n pieces $\gamma_{p_i p_{i+1}}$. In the planar limit $\nu_i := \frac{\hat{N}_i}{\hat{N}}$ is finite, and $\tilde{\mathcal{F}}(g_s, t, \nu_i) = \frac{1}{g_s^2} \tilde{\mathcal{F}}_0(t, \nu_i) + \ldots$. The saddle point approximation gives

$$
\tilde{\mathcal{F}}_0(t, \nu_i) = -t^2 S_{eff}(g_s = 0, t; s_a^{(j)*}(\nu_i)) \ , \tag{5.60}
$$

where (cf. (5.38)) $S_{eff}(g_s = 0, t; s_a^{(j)*}(\nu_i))$ is meant to be the value of $S(0, t; \lambda(s_a^{(j)*}(\nu_i)))$, with $\lambda(s_a^{(j)*}(\nu_i))$ the point on $\gamma_{p_i p_{i+1}}$ corresponding to the unique saddle point value $s_a^{(j)*}$ with fixed fraction ν_i of eigenvalues λ_m in D_i. Note that the $\frac{1}{\hat{N}^2} \sum \dot{\lambda}(s_m)$-term in (5.38) disappears in the present planar limit. One can go further and evaluate subleading terms. In particular, the remaining integral leads to the logarithm of the determinant of the $\hat{N} \times \hat{N}$-matrix of second derivatives of S at the saddle point. This is *not* order $g_s^0 \sim \hat{N}^0$ as naively expected from the expansion of $\tilde{\mathcal{F}}$ in powers of g_s^2. Instead one finds contributions like $-\hat{N} \log \hat{N}$, $\frac{\hat{N}}{2} \log t$. The point is that $-\hat{N}^2 S_{eff}(g_s, t, s^*(\nu))$ also has subleading contributions, which are dropped in the planar limit (5.60). In particular, one can check very explicitly for the Gaussian model that these subleading contributions in $S_{eff}(s^*)$ cancel the $\hat{N} \log \hat{N}$ and the $\frac{\hat{N}}{2} \log t$ pieces from the determinant. A remaining $\hat{N}$-dependent (but ν- and t-independent) constant could have been absorbed in the overall normalisation of Z. Hence the subleading terms are indeed $o(g_s^0) \sim o(\hat{N}^0)$.

It remains to sum over the $\hat{N}_i$ in (5.58). In the planar limit these sums are replaced

by integrals:

$$Z(g_s,t,J) = \int_0^1 \mathrm{d}\nu_1 \dots \int_0^1 \mathrm{d}\nu_n \, \delta\left(\sum_{i=1}^{n} \nu_i - 1\right) \exp\left[-\frac{1}{g_s^2}\left(t\sum_{i=1}^{n-1} J_i\nu_i - \tilde{\mathcal{F}}_0(t,\nu_i)\right) + c(\hat{N}) + o(g_s^0)\right] . \tag{5.61}$$

Once again, in the planar limit, this integral can be evaluated using the saddle point technique and for the source-dependent free energy $F(g_s,t,J) = \frac{1}{g_s^2}F_0(t,J) + \dots$ we find

$$F_0(t,J) = \sum_{i=1}^{n-1} J_i \; t\nu_i^* - \tilde{\mathcal{F}}_0(t,\nu_i^*) \ , \tag{5.62}$$

where ν_i^* solves the new saddle point equation,

$$tJ_i = \frac{\partial \tilde{\mathcal{F}}_0}{\partial \nu_i}(t,\nu_j) \ . \tag{5.63}$$

This shows that $F_0(t,J)$ is nothing but the Legendre transform of $\tilde{\mathcal{F}}_0(t,\nu_i^*)$ in the $n-1$ latter variables. If we define

$$S_i := t\nu_i^* \ , \text{ for } i = 1,\dots,n-1, \tag{5.64}$$

we have the inverse relation

$$S_i = \frac{\partial F_0}{\partial J_i}(t,J) \ , \tag{5.65}$$

and with $\mathcal{F}_0(t,S) := \tilde{\mathcal{F}}_0(t,\frac{S_i}{t})$, where $S := \{S_1,\dots,S_{n-1}\}$, one has from (5.62)

$$\mathcal{F}_0(t,S) = \sum_{i=1}^{n-1} J_iS_i - F_0(t,J) \ , \tag{5.66}$$

where J_i solves (5.65). From (5.60) and the explicit form of S_{eff}, Eq.(5.38) with $\hat{N} \to \infty$, we deduce that

$$\mathcal{F}_0(t,S) = t^2\mathcal{P}\int \mathrm{d}s \int \mathrm{d}s' \ \ln(\lambda(s){-}\lambda(s'))\rho_0(s;t,S_i)\rho_0(s';t,S_i){-}t\int \mathrm{d}s \, W(\lambda(s))\rho_0(s;t,S_i) \ , \tag{5.67}$$

where $\rho_0(s;t,S_i)$ is the eigenvalue density corresponding to the saddle point configuration $s_a^{(i)*}$ with $\frac{\hat{N}_i}{\hat{N}} = \nu_i$ fixed to be $\nu_i^* = \frac{S_i}{t}$. Hence it satisfies

$$t\int_{\gamma^{-1}(\mathcal{C}_i)} \rho_0(s;t,S_j)\mathrm{d}s = S_i \ \text{ for } i = 1,2,\dots n-1 \ , \tag{5.68}$$

and obviously

$$t\int_{\gamma^{-1}(\mathcal{C}_n)} \rho_0(s;t,S_j)\mathrm{d}s = t - \sum_{i=1}^{n-1} S_i \ . \tag{5.69}$$

Note that the integrals in (5.67) are convergent and $\mathcal{F}_0(t,S)$ is a well-defined function.

5.2 Special geometry relations

After this rather detailed study of the planar limit of holomorphic matrix models we now turn to the derivation of the special geometry relations for the Riemann surface (4.76) and hence the local Calabi-Yau (4.74). Recall that in the matrix model the $S_i = t\nu_i^*$ are real and therefore $\mathcal{F}_0(t,S)$ of Eq. (5.67) is a function of *real* variables. This is reflected by the fact that one can generate only a subset of all possible Riemann surfaces (4.76) from the planar limit of the holomorphic matrix model, namely those for which the $\nu_i^* = \frac{1}{4\pi i t}\int_{A^i}\zeta$ are real (recall $\zeta = y\mathrm{d}x$). We are, however, interested in the special geometry of the most general surface of the form (4.76), which can no longer be understood as a surface appearing in the planar limit of a matrix model. Nevertheless, for any such surface we can apply the formal construction of $\rho_0(s)$, which will in general be complex. Then one can use this complex "spectral density" to calculate the function $\mathcal{F}_0(t,S)$ from (5.67), that now depends on *complex* variables. Although this is *not* the planar limit of the free energy of the matrix model, it will turn out to be the prepotential for the general hyperelliptic Riemann surface (5.53) and hence of the local Calabi-Yau manifold (4.74).

5.2.1 Rigid special geometry

Let us then start from the general hyperelliptic Riemann surface (4.76) which we view as a two-sheeted cover of the complex plane (cf. Figs. 4.2, 4.1), with its cuts $\mathcal{C}_i$ between a_i^- and a_i^+. We choose a path γ on the upper sheet with parameterisation $\lambda(s)$ in such a way that $\mathcal{C}_i \subset \gamma$. The complex function $\rho_0(s)$ is determined from (5.47) and (5.52), as described above. We define the complex quantities

$$S_i := \frac{1}{4\pi i}\int_{A^i}\zeta = t\int_{\gamma^{-1}(\mathcal{C}_i)}\rho_0(s) \quad \text{for } i = 1,\dots,n-1\ , \tag{5.70}$$

and the prepotential $\mathcal{F}_0(t,S)$ as in (5.67) (of course, t is $-\frac{1}{4}$ times the leading coefficient of f_0 and it can now be complex as well).

Following [95] one defines the "principal value of y_0" along the path γ (c.f. (5.40))

$$y_0^p(s) := \frac{1}{2}\lim_{\epsilon\to 0}[y_0(\lambda(s)+i\epsilon\dot{\lambda}(s)) + y_0(\lambda(s)-i\epsilon\dot{\lambda}(s))]\ . \tag{5.71}$$

For points $\lambda(s)\in\gamma$ outside $\mathcal{C} := \cup_i\mathcal{C}_i$ we have $y_0^p(s) = y_0(\lambda(s))$, while $y_0^p(s) = 0$ on $\mathcal{C}$. With

$$\phi(s) := W(\lambda(s)) - 2t\mathcal{P}\int \mathrm{d}s'\ln(\lambda(s)-\lambda(s'))\rho_0(s';t,S_i) \tag{5.72}$$

one finds, using (5.46), (5.52) and (5.40),

$$\frac{d}{ds}\phi(s) = \dot{\lambda}(s)y_0^p(s)\ . \tag{5.73}$$

The fact that $y_0^p(s)$ vanishes on $\mathcal{C}$ implies

$$\phi(s) = \xi_i := \text{constant on } \mathcal{C}_i\ . \tag{5.74}$$

Integrating (5.73) between $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ gives

$$\xi_{i+1} - \xi_i = \int_{a_i^+}^{a_{i+1}^-} \mathrm{d}\lambda\; y_0(\lambda) = \frac{1}{2}\int_{\beta_i} \mathrm{d}x\; y(x) = \frac{1}{2}\int_{\beta_i} \zeta\ . \tag{5.75}$$

From (5.67) we find for $i < n$

$$\frac{\partial}{\partial S_i}\mathcal{F}_0(t,S) = -t\int \mathrm{d}s \frac{\partial \rho_0(s;t,S_j)}{\partial S_i}\phi(s) = \xi_n - \xi_i\ . \tag{5.76}$$

To arrive at the last equality we used that $\rho_0(s) \equiv 0$ on the complement of the cuts, while on the cuts $\phi(s)$ is constant and we can use (5.68) and (5.69). Then, for $i < n-1$,

$$\frac{\partial}{\partial S_i}\mathcal{F}_0(t,S) - \frac{\partial}{\partial S_{i+1}}\mathcal{F}_0(t,S) = \frac{1}{2}\int_{\beta_i} \zeta\ . \tag{5.77}$$

For $i = n-1$, on the other hand, we find

$$\frac{\partial}{\partial S_{n-1}}\mathcal{F}_0(t,S) = \xi_n - \xi_{n-1} = \frac{1}{2}\int_{\beta_{n-1}} \zeta\ . \tag{5.78}$$

We change coordinates to

$$\tilde{S}_i := \sum_{j=1}^{i} S_j\ , \tag{5.79}$$

and find the rigid special geometry[7] relations

$$\tilde{S}_i = \frac{1}{4\pi i}\int_{\alpha_i} \zeta\ , \tag{5.80}$$

$$\frac{\partial}{\partial \tilde{S}_i}\mathcal{F}_0(t,\tilde{S}) = \frac{1}{2}\int_{\beta_i} \zeta\ . \tag{5.81}$$

for $i = 1, \dots, n-1$. Note that the basis of one-cycles that appears in these equations is the one shown in Fig. 4.1 and differs from the one used in [95]. The origin of this difference is the fact that we introduced only $n-1$ currents J_i in the partition function (5.57).

Next we use the same methods to derive the relation between the integrals of ζ over the cycles $\hat{\alpha}$ and $\hat{\beta}$ and the planar free energy [P5].

5.2.2 Integrals over relative cycles

The first of these integrals encircles all the cuts, and by deforming the contour one sees that it is given by the residue of the pole of ζ at infinity, which is determined by the leading coefficient of $f_0(x)$:

$$\frac{1}{4\pi i}\int_{\hat{\alpha}} \zeta = t\ . \tag{5.82}$$

[7] For a review of rigid special geometry see appendix C.2.

The cycle $\hat{\beta}$ starts at infinity of the lower sheet, runs to the n-th cut and from there to infinity on the upper sheet. The integral of ζ along $\hat{\beta}$ is divergent, so we introduce a (real) cut-off Λ_0 and instead take $\hat{\beta}$ to run from Λ_0' on the lower sheet through the n-th cut to Λ_0 on the upper sheet. We find

$$\begin{aligned}\frac{1}{2}\int_{\hat{\beta}}\zeta = \int_{a_n^+}^{\Lambda_0} y_0(\lambda)\mathrm{d}\lambda &= \phi(\lambda^{-1}(\Lambda_0)) - \phi(\lambda^{-1}(a_n^+)) = \phi(\lambda^{-1}(\Lambda_0)) - \xi_n \\ &= W(\Lambda_0) - 2t\mathcal{P}\int \mathrm{d}s' \ln(\Lambda_0 - \lambda(s'))\rho_0(s';t,\tilde{S}_i) - \xi_n \ .\end{aligned} \tag{5.83}$$

On the other hand we can calculate

$$\frac{\partial}{\partial t}\mathcal{F}_0(t,\tilde{S}) = -\int \mathrm{d}s\ \phi(\lambda(s))\frac{\partial}{\partial t}[t\rho_0(s;t,\tilde{S}_i)] = -\sum_{i=1}^{n}\xi_i\int_{\gamma^{-1}(\mathcal{C}_i)}\mathrm{d}s\ \frac{\partial}{\partial t}[t\rho_0(s;t,\tilde{S}_i)] = -\xi_n\ , \tag{5.84}$$

(where we used (5.68) and (5.69)) which leads to

$$\begin{aligned}\frac{1}{2}\int_{\hat{\beta}}\zeta &= \frac{\partial}{\partial t}\mathcal{F}_0(t,\tilde{S}) + W(\Lambda_0) - 2t\mathcal{P}\int \mathrm{d}s' \ln(\Lambda_0 - \lambda(s'))\rho_0(s';t,\tilde{S}_i) \\ &= \frac{\partial}{\partial t}\mathcal{F}_0(t,\tilde{S}) + W(\Lambda_0) - t\log\Lambda_0^2 + o\left(\frac{1}{\Lambda_0}\right)\ .\end{aligned} \tag{5.85}$$

Together with (5.82) this looks very similar to the usual special geometry relation. In fact, the cut-off independent term is the one one would expect from special geometry. However, the equation is corrected by cut-off dependent terms. The last terms vanishes if we take Λ_0 to infinity but there remain two divergent terms which we are going to study in detail below.[8] For a derivation of (5.85) in a slightly different context see [29].

5.2.3 Homogeneity of the prepotential

The prepotential on the moduli space of complex structures of a *compact* Calabi-Yau manifold is a holomorphic function that is homogeneous of degree two. On the other hand, the structure of the local Calabi-Yau manifold (4.74) is captured by a Riemann surface and it is well-known that these are related to rigid special geometry. The prepotential of rigid special manifolds does not have to be homogeneous (see for example [38]), and it is therefore important to explore the homogeneity structure of $\mathcal{F}_0(t,\tilde{S})$. The result is quite interesting and it can be written in the form

$$\sum_{i=1}^{n-1}\tilde{S}_i\frac{\partial\mathcal{F}_0}{\partial\tilde{S}_i}(t,\tilde{S}_i) + t\frac{\partial\mathcal{F}_0}{\partial t}(t,\tilde{S}_i) = 2\mathcal{F}_0(t,\tilde{S}_i) + t\int \mathrm{d}s\ \rho_0(s;t,\tilde{S}_i)W(\lambda(s))\ . \tag{5.86}$$

[8] Of course, one could define a cut-off dependent function $\mathcal{F}^{\Lambda_0}(t,\tilde{S}) := \mathcal{F}_0(t,\tilde{S}) + tW(\Lambda_0) - \frac{t^2}{2}\log\Lambda_0^2$ for which one has $\frac{1}{2}\int_{\hat{\beta}}\zeta = \frac{\partial\mathcal{F}^{\Lambda_0}}{\partial t} + o\left(\frac{1}{\Lambda_0}\right)$ similar to [40]. Note, however, that this is not a standard special geometry relation due to the presence of the $o\left(\frac{1}{\Lambda_0}\right)$-terms. Furthermore, $\mathcal{F}^{\Lambda_0}$ has no interpretation in the matrix model and is divergent as $\Lambda_0 \to \infty$.

To derive this relation we rewrite Eq. (5.67) as

$$\begin{aligned} 2\mathcal{F}_0(t,\tilde{S}_i) &= -t\int \mathrm{d}s\ \rho_0(s;t,\tilde{S}_i)\,[\phi(s)+W(\lambda(s))] \\ &= -t\int \mathrm{d}s\ \rho_0(s;t,\tilde{S}_i)W(\lambda(s)) + \sum_{i=1}^{n-1}(\xi_n-\xi_i)S_i - t\xi_n\ . \end{aligned} \tag{5.87}$$

Furthermore, we have $\sum_{i=1}^{n-1}\tilde{S}_i\frac{\partial\mathcal{F}_0}{\partial\tilde{S}_i}(t,\tilde{S}_i) = \sum_{i=1}^{n-1}S_i\frac{\partial\mathcal{F}_0}{\partial S_i}(t,S_i) = \sum_{i=1}^{n-1}S_i(\xi_n-\xi_i)$, where we used (5.76). The result then follows from (5.84).
Of course, the prepotential was not expected to be homogeneous, since already for the simplest example, the conifold, $\mathcal{F}_0$ is known to be non-homogeneous (see section 5.2.5). However, Eq. (5.86) shows the precise way in which the homogeneity relation is modified on the local Calabi-Yau manifold (4.74).

5.2.4 Duality transformations

The choice of the basis $\{\alpha^i,\beta_j,\hat{\alpha},\hat{\beta}\}$ for the (relative) one-cycles on the Riemann surface was particularly useful in the sense that the integrals over the compact cycles α^i and β_j reproduce the familiar rigid special geometry relations, whereas new features appear only in the integrals over $\hat{\alpha}$ and $\hat{\beta}$. In particular, we may perform any symplectic transformation of the compact cycles α^i and β_j, $i,j=1,\dots n-1$, among themselves to obtain a new set of compact cycles which we call a^i and b_j. Such symplectic transformations can be generated from (i) transformations that do not mix a-type and b-type cycles, (ii) transformations $a^i=\alpha^i$, $b_i=\beta_i+\alpha^i$ for some i and (iii) transformations $a^i=\beta_i$, $b_i=-\alpha^i$ for some i. (These are analogue to the trivial, the T and the S modular transformations of a torus.) For transformations of the first type the prepotential $\mathcal{F}$ remains unchanged, except that it has to be expressed in terms of the new variables s_i, which are the integrals of ζ over the new a^i cycles. Since the transformation is symplectic, the integrals over the new b_j cycles then automatically are the derivatives $\frac{\partial\mathcal{F}_0(t,s)}{\partial s_i}$. For transformations of the second type the new prepotential is given by $\mathcal{F}_0(t,\tilde{S}_i)+i\pi\tilde{S}_i^2$ and for transformations of the third type the prepotential is a Legendre transform with respect to $\int_{a^i}\zeta$. In the corresponding gauge theory the latter transformations realise electric-magnetic duality. Consider e.g. a symplectic transformation that exchanges all compact α^i-cycles with all compact β_i cycles:

$$\begin{pmatrix}\alpha^i\\ \beta_i\end{pmatrix} \to \begin{pmatrix}a^i\\ b_i\end{pmatrix} = \begin{pmatrix}\beta_i\\ -\alpha^i\end{pmatrix},\quad i=1,\dots n-1\ . \tag{5.88}$$

Then the new variables are the integrals over the a_i-cycles which are

$$\tilde{\pi}_i := \frac{1}{2}\int_{\beta_i}\zeta = \frac{\partial\mathcal{F}_0(t,\tilde{S})}{\partial\tilde{S}_i} \tag{5.89}$$

and the dual prepotential is given by the Legendre transformation

$$\mathcal{F}_D(t,\tilde{\pi}) := \sum_{i=1}^{n-1}\tilde{S}_i\tilde{\pi}_i - \mathcal{F}_0(t,\tilde{S})\ , \tag{5.90}$$

such that the new special geometry relation is

$$\frac{\partial \mathcal{F}_D(t,\tilde{\pi})}{\partial \tilde{\pi}_i} = \tilde{S}_i = \frac{1}{4\pi i}\int_{\alpha^i} \zeta \ . \tag{5.91}$$

Comparing with (5.62) one finds that $\mathcal{F}_D(t,\tilde{\pi})$ actually coincides with $F_0(t,J)$ where $J_i - J_{i+1} = \tilde{\pi}_i$ for $i = 1, \ldots n-2$ and $J_{n-1} = \tilde{\pi}_{n-1}$.

Next, let us see what happens if we also include symplectic transformations involving the relative cycles $\hat{\alpha}$ and $\hat{\beta}$. An example of a transformation of type (i) that does not mix $\{\alpha, \hat{\alpha}\}$ with $\{\beta, \hat{\beta}\}$ cycles is the one from $\{\alpha^i, \beta_j, \hat{\alpha}, \hat{\beta}\}$ to $\{A^i, B_j\}$, c.f. Figs. 4.1, 4.2. This corresponds to

$$\begin{aligned} \bar{S}_1 &:= \tilde{S}_1 \ , \\ \bar{S}_i &:= \tilde{S}_i - \tilde{S}_{i-1} \quad \text{for} \quad i = 2, \ldots n-1 \ , \\ \bar{S}_n &:= t - \tilde{S}_{n-1} \ , \end{aligned} \tag{5.92}$$

so that

$$\bar{S}_i = \frac{1}{4\pi i}\int_{A^i} \zeta \ . \tag{5.93}$$

The prepotential does not change, except that it has to be expressed in terms of the $\bar{S}_i$. One then finds for $B_i = \sum_{j=i}^{n-1} \beta_j + \hat{\beta}$

$$\frac{1}{2}\int_{B_i} \zeta = \frac{\partial \mathcal{F}_0(\bar{S})}{\partial \bar{S}_i} + W(\Lambda_0) - \left(\sum_{j=1}^{n} \bar{S}_i\right) \log \Lambda_0^2 + o\left(\frac{1}{\Lambda_0}\right) \ . \tag{5.94}$$

We see that as soon as one "mixes" the cycle $\hat{\beta}$ into the set $\{\beta_i\}$ one obtains a number of relative cycles B_i for which the special geometry relations are corrected by cut-off dependent terms. An example of transformation of type (iii) is $\hat{\alpha} \to \hat{\beta}$, $\hat{\beta} \to -\hat{\alpha}$. Instead of t one then uses

$$\hat{\pi} := \frac{\partial \mathcal{F}_0(t,\tilde{S})}{\partial t} = \lim_{\Lambda_0 \to \infty} \left[\frac{1}{2}\int_{\hat{\beta}} \zeta - W(\Lambda_0) + t \log \Lambda_0^2\right] \tag{5.95}$$

as independent variable and the Legendre transformed prepotential is

$$\hat{\mathcal{F}}(\hat{\pi}, \tilde{S}) := t\hat{\pi} - \mathcal{F}_0(t, \tilde{S}) \ , \tag{5.96}$$

so that now

$$\frac{\partial \hat{\mathcal{F}}(\hat{\pi}, \tilde{S})}{\partial \hat{\pi}} = t = \frac{1}{4\pi i}\int_{\hat{\alpha}} \zeta \ . \tag{5.97}$$

Note that the prepotential is well-defined and independent of the cut-off in all cases (in contrast to the treatment in [40]). The finiteness of $\hat{\mathcal{F}}$ is due to $\hat{\pi}$ being the *corrected, finite* integral over the relative $\hat{\beta}$-cycle.

Note also that if one exchanges all coordinates simultaneously, i.e. $\alpha^i \to \beta_i$, $\hat{\alpha} \to \hat{\beta}, \beta_i \to -\alpha^i$, $\hat{\beta} \to -\hat{\alpha}$, one has

$$\hat{\mathcal{F}}_D(\hat{\pi}, \tilde{\pi}) := t\hat{\pi} + \sum_{i=1}^{n-1} \tilde{S}_i \tilde{\pi}_i - \mathcal{F}_0(t, \tilde{S}_i) \ . \tag{5.98}$$

Using the generalised homogeneity relation (5.86) this can be rewritten as

$$\hat{\mathcal{F}}_D(\hat{\pi}, \tilde{\pi}) = \mathcal{F}_0(t, \tilde{S}_i) + t \int \mathrm{d}s \ \rho_0(s; t, \tilde{S}_i) W(\lambda(s)) \ . \tag{5.99}$$

It would be quite interesting to understand the results of this chapter concerning the parameter spaces of local Calabi-Yau manifolds in a more geometrical way in the context of (rigid) special Kähler manifolds along the lines of C.2.

5.2.5 Example and summary

Let us pause for a moment and collect the results that we have deduced so far. In order to compute the effective superpotential of our gauge theory we have to study the integrals of Ω over all compact and non-compact three-cycles on the space X_{def}, given by

$$W'(x)^2 + f_0(x) + v^2 + w^2 + z^2 = 0 \ . \tag{5.100}$$

These integrals map to integrals of $y\mathrm{d}x$ on the Riemann surface Σ, given by $y^2 = W'(x)^2 + f_0(x)$ over all the elements of $H_1(\Sigma, \{Q, Q'\})$. We have shown that it is useful to split the elements of this set into a set of compact cycles α^i and β_i and a set containing the compact cycle $\hat{\alpha}$ and the non-compact cycle $\hat{\beta}$, which together form a symplectic basis. The corresponding three-cycles on the Calabi-Yau manifold are $\Gamma_{\alpha^i}, \Gamma_{\beta_j}, \Gamma_{\hat{\alpha}}, \Gamma_{\hat{\beta}}$. This choice of cycles is appropriate, since the properties that arise from the non-compactness of the manifold are then captured entirely by the integral of the holomorphic three-form Ω over the non-compact three-cycle $\Gamma_{\hat{\beta}}$ which corresponds to $\hat{\beta}$. Indeed, combining (4.79), (5.80), (5.81), (5.82) and (5.85) one finds the following relations

$$-\frac{1}{2\pi i}\int_{\Gamma_{\alpha^i}} \Omega = 2\pi i \tilde{S}_i \ , \tag{5.101}$$

$$-\frac{1}{2\pi i}\int_{\Gamma_{\beta_i}} \Omega = \frac{\partial \mathcal{F}_0(t, \tilde{S})}{\partial \tilde{S}_i} \ , \tag{5.102}$$

$$-\frac{1}{2\pi i}\int_{\Gamma_{\hat{\alpha}}} \Omega = 2\pi i t \ , \tag{5.103}$$

$$-\frac{1}{2\pi i}\int_{\Gamma_{\hat{\beta}}} \Omega = \frac{\partial \mathcal{F}_0(t, \tilde{S})}{\partial t} + W(\Lambda_0) - t\log\Lambda_0^2 + o\left(\frac{1}{\Lambda_0}\right) \ . \tag{5.104}$$

These relations are useful to calculate the integrals, since $\mathcal{F}_0(t, \tilde{S})$ is the (Legendre transform of) the free energy coupled to sources, and it can be evaluated from (5.67).

In the last relation the integral is understood to be over the regulated cycle $\Gamma_{\hat{\beta}}$ which is an S^2-fibration over a line segment running from the n-th cut to the cut-off Λ_0. Clearly, once the cut-off is removed, the last integral diverges. This divergence will be studied in more detail below.

The conifold

Let us illustrate these ideas by looking at the simplest example, the deformed conifold. In this case we have $n = 1$, $W(x) = \frac{x^2}{2}$ and $f_0(x) = -\mu = -4t$, $\mu \in \mathbb{R}^+$, and $X_{def} = C_{def}$ is given by

$$x^2 + v^2 + w^2 + z^2 - \mu = 0 \ . \tag{5.105}$$

As $n = 1$ the corresponding Riemann surface has genus zero. Then

$$\zeta = y\mathrm{d}x = \left\{ \begin{array}{ll} \sqrt{x^2 - 4t}\ \mathrm{d}x & \text{on the upper sheet} \\ -\sqrt{x^2 - 4t}\ \mathrm{d}x & \text{on the lower sheet} \end{array} \right. . \tag{5.106}$$

We have a cut $\mathcal{C} = [-2\sqrt{t}, 2\sqrt{t}]$ and take $\lambda(s) = s$ to run along the real axis. The corresponding $\rho_0(s)$ is immediately obtained from (5.47) and (5.52) and yields the well-known $\rho_0(s) = \frac{1}{2\pi t}\sqrt{4t - s^2}$, for $s \in [-2\sqrt{t}, 2\sqrt{t}]$ and zero otherwise, and from (5.67) we find the planar free energy

$$\mathcal{F}_0(t) = \frac{t^2}{2}\log t - \frac{3}{4}t^2 \ . \tag{5.107}$$

Note that $t \int \mathrm{d}s\, \rho_0(s) W(\lambda(s)) = \frac{t^2}{2}$ and $\mathcal{F}_0$ satisfies the generalised homogeneity relation (5.86)

$$t\frac{\partial \mathcal{F}_0}{\partial t}(t) = 2\mathcal{F}_0(t) + \frac{t^2}{2} \ . \tag{5.108}$$

For the deformed conifold the integrals over Ω can be calculated without much difficulty and one obtains

$$-\frac{1}{2\pi i}\int_{\Gamma_{\hat{\alpha}}} \Omega = \frac{1}{2}\int_{\hat{\alpha}} \zeta \ = \ 2\pi i t = 2\pi i \bar{S} \tag{5.109}$$

$$-\frac{1}{2\pi i}\int_{\Gamma_{\hat{\beta}}} \Omega = \frac{1}{2}\int_{\hat{\beta}} \zeta \ = \ \frac{\Lambda_0}{2}\sqrt{\Lambda_0^2 - 4t} - 2t\log\left(\frac{\Lambda_0}{2\sqrt{t}} + \sqrt{\frac{\Lambda_0^2}{4t} - 1}\right) \ . \tag{5.110}$$

On the other hand, using the explicit form of $\mathcal{F}_0(t)$ we find

$$\frac{\partial \mathcal{F}_0(t)}{\partial t} + W(\Lambda_0) - t\log\Lambda_0^2 = \frac{\Lambda_0^2}{2} + t\log\left(\frac{t}{\Lambda_0^2}\right) - t \tag{5.111}$$

which agrees with (5.110) up to terms of order $o\left(\frac{1}{\Lambda_0^2}\right)$.

Chapter 6

Superstrings, the Geometric Transition and Matrix Models

We are now in a position to combine all the results obtained so far, and explain the conjecture of Cachazo, Intriligator and Vafa [27] in more detail. Recall that our goal is to determine the effective superpotential, and hence the vacuum structure, of $\mathcal{N}=1$ super Yang-Mills theory with gauge group $U(N)$ coupled to a chiral superfield Φ in the adjoint representation with tree-level superpotential

$$W(\Phi)=\sum_{k=1}^{\infty}\frac{g_k}{k}\operatorname{tr}\Phi^k+g_0\ . \tag{6.1}$$

In a given vacuum of this theory the eigenvalues of Φ sit at the critical points of $W(x)$, and the gauge group $U(N)$ breaks to $\prod_{i=1}^{n}U(N_i)$, if N_i is the number of eigenvalues at the i-th critical point of $W(x)$. Note that such a Φ also satisfies the D-flatness condition (3.21), since $\operatorname{tr}([\Phi^\dagger,\Phi]^2)=0$.

At low energies the $SU(N_i)$-part of the group $U(N_i)$ confines and the remaining gauge group is $U(1)^n$. The good degrees of freedom in this low energy limit are the massless photons of the $U(1)\subset U(N_i)$ and the massive chiral superfields[1] S_i^{gt} with the gaugino bilinear $\lambda_a^{(i)}\lambda^{a(i)}$ as their lowest component. We want to determine the quantities $\langle S_i^{gt}\rangle$, and these can be obtained from minimising the effective superpotential,

$$\left.\frac{\partial W_{eff}^{gt}(S_i^{gt})}{\partial S_i^{gt}}\right|_{\langle S_i^{gt}\rangle}=0\ . \tag{6.2}$$

Therefore, we are interested in determining the function $W_{eff}^{gt}(S_i^{gt})$. Since we are in the low energy regime of the gauge theory, where the coupling constant is large, this is a very hard problem in field theory. However, it turns out that the effective superpotential can be calculated in a very elegant way in the context of string theory.

[1]For clarity we introduce a superscript gt for gauge theory quantities and st for string theory objects. These will be identified momentarily, and then the superscript will be dropped.

As mentioned in the introduction, the specific vacuum of the original $U(N)$ gauge theory, in which the gauge group $U(N)$ is broken to $\prod_{i=1}^n U(N_i)$, can be generated from type IIB theory by wrapping N_i D5-branes around the i-th $\mathbb{CP}^1$ in X_{res} (as defined in (4.71)). We will not give a rigorous proof of this statement, but refer the reader to [87] where many of the details have been worked out. Here we only try to motivate the result, using some more or less heuristic arguments. Clearly, the compactification of type IIB on X_{res} leads to an $\mathcal{N}=2$ theory in four dimensions. The geometric engineering of $\mathcal{N}=2$ theories from local Calabi-Yau manifolds is reviewed in [105]. Introducing the $N=\sum_i N_i$ D5-branes now has two effects. First, the branes reduce the amount of supersymmetry. If put at arbitrary positions, the branes will break supersymmetry completely. However, as shown in [18], if two dimensions of the branes wrap holomorphic cycles in X_{res}, which are nothing but the resolved $\mathbb{CP}^1$, and the other dimensions fill Minkowski space, they only break half of the supersymmetry, leading to an $\mathcal{N}=1$ theory. Furthermore, $U(N_i)$ vector multiplets arise from open strings polarised along Minkowski space, whereas those strings that start and end on the wrapped branes lead to a four-dimensional chiral superfield Φ in the adjoint representation of the gauge group. A brane wrapped around a holomorphic cycle $\mathcal{C}$ in a Calabi-Yau manifold X can be deformed without breaking the supersymmetry, provided there exist holomorphic sections of the normal bundle $N\mathcal{C}$. These deformations are the scalar fields in the chiral multiplet, which therefore describe the position of the wrapped brane. The number of these deformations, and hence of the chiral fields, is therefore given by the number of holomorphic sections of $N\mathcal{C}$. However, on some geometries these deformations are obstructed, i.e. one cannot construct a finite deformation from an infinitesimal one (see [87] for the mathematical details). These obstructions are reflected in the fact that there exists a superpotential for the chiral superfield at the level of the four-dimensional action. In [87] it is shown that the geometry of X_{res} is such that the superpotential of the four-dimensional field theory is nothing but $W(\Phi)$.

We have seen that the local Calabi-Yau manifold X_{res} can go through a geometric transition, leading to the deformed space X_{def} (as defined in (4.74)). This tells us that there is another interesting setup, which is intimately linked to the above, namely type IIB on X_{def} with additional three-form background flux G_3. The background flux is necessary in order to ensure that the four-dimensional theory is $\mathcal{N}=1$ supersymmetric. A heuristic argument for its presence has been given in the introduction. The $\mathcal{N}=1$ four-dimensional theory generated by this string theory then contains n $U(1)$ vector superfields and n chiral superfields[2] $\bar{S}_i^{st}$, the lowest component of which is proportional to the size[3] of the three-cycles in X_{def},

$$\bar{S}_i^{st} := \frac{1}{4\pi^2}\int_{\Gamma_{A^i}} \Omega \ . \tag{6.3}$$

[2]The bar in $\bar{S}_i^{st}$ indicates that in the defining equation (6.3) one uses the set of cycles $\Gamma_{A^i}, \Gamma_{B_j}$, c.f. Eq. (5.92).

[3]Ω is a calibration, i.e. it reduces to the volume form on Γ_{A^i}, see e.g. [86].

The superpotential W_{eff}^{st} for these fields is given by the formula[4] [75]

$$W_{eff}^{st}(\bar{S}_i^{st}) = \frac{1}{2\pi i}\sum_{i=1}^{n}\left(\int_{\Gamma_{A_i}} G_3 \int_{\Gamma_{B_i}} \Omega - \int_{\Gamma_{B_i}} G_3 \int_{\Gamma_{A^i}} \Omega\right) \ , \tag{6.4}$$

see also [128], [106]. Of course, W_{eff}^{st} does not only depend on the $\bar{S}_i^{st}$, but also on the g_k, since Ω is defined in terms of the tree-level superpotential, c.f. Eq. (4.75). Furthermore, as we will see, it also depends on parameters Λ_i, which will be identified with the dynamical scales of the $SU(N_i)$ theories below. It is quite interesting to compare the derivation of this formula in [75] with the one of the Veneziano-Yankielowicz formula in [132]. The logic is very similar, and indeed, as we will see, Eq. (6.4) gives the effective superpotential in the Veneziano-Yankielowicz sense. It is useful to define

$$\tilde{S}_i^{st} := \frac{1}{4\pi^2}\int_{\Gamma_{\alpha^i}} \Omega \quad , \quad t^{st} := \frac{1}{4\pi^2}\int_{\Gamma_{\hat{\alpha}}} \Omega \ , \tag{6.5}$$

for $i = 1, \dots n-1$, and use the set of cycles $\{\Gamma_{\alpha^i}, \Gamma_{\beta_j}, \Gamma_{\hat{\alpha}}, \Gamma_{\hat{\beta}}\}$ instead (see the discussion in section 4.2.3 and Figs. 4.2, 4.1 for the definition of these cycles). Then we find

$$\begin{aligned} & W_{eff}^{st}(t^{st}, \tilde{S}_i^{st}) = \\ & = \frac{1}{2\pi i}\sum_{i=1}^{n-1}\left(\int_{\Gamma_{\alpha^i}} G_3 \int_{\Gamma_{\beta_i}} \Omega - \int_{\Gamma_{\beta_i}} G_3 \int_{\Gamma_{\alpha^i}} \Omega\right) + \frac{1}{2\pi i}\left(\int_{\Gamma_{\hat{\alpha}}} G_3 \int_{\Gamma_{\hat{\beta}}} \Omega - \int_{\Gamma_{\hat{\beta}}} G_3 \int_{\Gamma_{\hat{\alpha}}} \Omega\right) \\ & = -\frac{1}{2}\sum_{i=1}^{n-1}\left(\int_{\Gamma_{\alpha^i}} G_3 \int_{\beta_i} \zeta - \int_{\Gamma_{\beta_i}} G_3 \int_{\alpha^i} \zeta\right) - \frac{1}{2}\left(\int_{\Gamma_{\hat{\alpha}}} G_3 \int_{\hat{\beta}} \zeta - \int_{\Gamma_{\hat{\beta}}} G_3 \int_{\hat{\alpha}} \zeta\right) \ . \end{aligned} \tag{6.6}$$

Here we used the fact that, as explained in section 4.2.3, the integrals of Ω over three-cycles reduce to integrals of $\zeta := y\mathrm{d}x$ over the one-cycles in $H_1(\Sigma, \{Q, Q'\})$ on the Riemann surface Σ,

$$y^2 = W'(x)^2 + f_0(x) \ , \tag{6.7}$$

c.f. Eq. (4.79). However, from our analysis of the matrix model we know that the integral of ζ over the cycle $\hat{\beta}$ is divergent, c.f. Eq. (5.85). Since the effective superpotential has to be finite we see that (6.6) cannot yet be the correct formula. Indeed from inspection of (5.85) we find that it contains two divergent terms, one logarithmic, and one polynomial divergence. The way how these divergences are dealt with, and how a finite effective superpotential is generated, will be explained in the first section of this chapter.

[4] The original formula has the elegant form $W_{eff}^{st} = \frac{1}{2\pi i}\int_X G_3 \wedge \Omega$, which can then be written as (6.4) using the Riemann bilinear relations for Calabi-Yau manifolds. However, X_{def} is a *local* Calabi-Yau manifold and we are not aware of a proof that the bilinear relation does hold for these spaces as well.

The claim of Cachazo, Intriligator and Vafa is that the four-dimensional superpotential (6.4), generated from IIB on $\mathbb{R}^4 \times X_{def}$ with background flux, is nothing but the effective superpotential (in the Veneziano-Yankiwlowicz sense) of the original gauge theory,

$$W^{gt}_{eff}(S^{gt}_i) \equiv W^{st}_{eff}(\bar{S}^{st}_i) \; . \tag{6.8}$$

Put differently we have

$$\langle S^{gt}_i \rangle = \langle \bar{S}^{st}_i \rangle \; , \tag{6.9}$$

where the left-hand side is the vacuum expectation value of the gauge theory operator S^{gt}_i, whereas the right-hand side is a Kähler parameter of X_{def} (proportional to the size of Γ_{A^i}), that solves

$$\left. \frac{\partial W^{st}_{eff}(\bar{S}^{st}_i)}{\partial \bar{S}^{st}_i} \right|_{\langle \bar{S}^{st}_i \rangle} = 0 \; . \tag{6.10}$$

Therefore, the vacuum structure of a gauge theory can be studied by evaluating W^{st}_{eff}, i.e. by performing integrals in the geometry X_{def}. From now on the superscripts gt and st will be suppressed.

We are going to check the conjecture of Cachazo-Intriligator and Vafa by looking at simple examples below.

6.1 Superpotentials from string theory with fluxes

In this section we will first analyse the divergences in (6.6) and show that the effective potential is actually finite if we modify the integration over Ω in a suitable way. We closely follow the analysis of [P5], where the necessary correction terms were calculated. Then we use our matrix model results to relate the effective superpotential to the planar limit of the matrix model free energy.

6.1.1 Pairings on Riemann surfaces with marked points

In order to understand the divergences somewhat better, we will study the meromorphic one-form $\zeta := y\mathrm{d}x$ on the Riemann surface Σ given by Eq. (6.7) in more detail. Recall that Q, Q' are those points on the Riemann surface that correspond to ∞, ∞' on the two-sheets of the representation (6.7) and that these are the points where ζ has a pole. The surface with the points Q, Q' removed is denoted by $\hat{\Sigma}$. First of all we observe that the integrals $\int_{\alpha^i} \zeta$ and $\int_{\beta_j} \zeta$ only depend on the cohomology class $[\zeta] \in H^1(\hat{\Sigma})$, whereas $\int_{\hat{\beta}} \zeta$ (where $\hat{\beta}$ extends between the poles of ζ, i.e. from ∞' on the lower sheet, corresponding to Q', to ∞ on the upper sheet, corresponding to Q,) is not only divergent, it also depends on the representative of the cohomology class, since for $\tilde{\zeta} = \zeta + \mathrm{d}\rho$ one has $\int_{\hat{\beta}} \tilde{\zeta} = \int_{\hat{\beta}} \zeta + \int_{\partial \hat{\beta}} \rho \left(\neq \int_{\hat{\beta}} \zeta \right)$. Note that the integral would be independent of the choice of the representative if we constrained ρ to be zero at $\partial \hat{\beta}$.

But as we marked Q, Q' on the Riemann surface, ρ is allowed to take finite or even infinite values at these points and therefore the integrals differ in general.

The origin of this complication is, of course, that our cycles are elements of the *relative* homology group $H_1(\Sigma, \{Q, Q'\})$. Then, their is a natural map $\langle .,. \rangle : H_1(\Sigma, \{Q, Q'\}) \times H^1(\Sigma, \{Q, Q'\}) \to \mathbb{C}$. $H^1(\Sigma, \{Q, Q'\})$ is the relative cohomology group dual to $H_1(\Sigma, \{Q, Q'\})$. In general, on a manifold M with submanifold N, elements of relative cohomology can be defined as follows (see for example [88]). Let $\Omega^k(M, N)$ be the set of k-forms on M that vanish on N. Then $H^k(M, N) := Z^k(M, N)/B^k(M, N)$, where $Z^k(M, N) := \{\omega \in \Omega^k(M, N) : \mathrm{d}\omega = 0\}$ and $B^k(M, N) := \mathrm{d}\Omega^{k-1}(M, N)$. For $[\hat{\Gamma}] \in H_k(M, N)$ and $[\eta] \in H^k(M, N)$ the pairing is defined as

$$\langle \hat{\Gamma}, \eta \rangle_0 := \int_{\hat{\Gamma}} \eta \ . \tag{6.11}$$

This does not depend on the representative of the classes, since the forms are constraint to vanish on N.

Now consider $\xi \in \Omega^k(M)$ such that $i^*\xi = \mathrm{d}\phi$, where $i : N \to M$ is the inclusion mapping. Note that ξ is not a representative of an element of relative cohomology, as it does not vanish on N. However, there is another representative in its cohomology class $[\xi] \in H^k(M)$, namely $\xi_\phi = \xi - \mathrm{d}\phi$ which now is also a representative of $H^k(M, N)$. For elements ξ with this property we can extend the definition of the pairing to

$$\langle \hat{\Gamma}, \xi \rangle_0 := \int_{\hat{\Gamma}} (\xi - \mathrm{d}\phi) \ . \tag{6.12}$$

More details on the various possible definitions of relative (co-)homology can be found in appendix B.3.

Clearly, the one-form $\zeta = y\mathrm{d}x$ on $\hat{\Sigma}$ is not a representative of an element of $H^1(\Sigma, \{Q, Q'\})$. According to the previous discussion, one might try to find $\zeta_\varphi = \zeta - \mathrm{d}\varphi$ where φ is chosen in such a way that ζ_φ vanishes at Q, Q', so that in particular $\int_{\hat{\beta}} \zeta_\varphi =$ finite. In other words, we would like to find a representative of $[\zeta] \in H^1(\hat{\Sigma})$ which is also a representative of $H^1(\Sigma, \{Q, Q'\})$. Unfortunately, this is not possible, because of the logarithmic divergence, i.e. the simple poles at Q, Q', which cannot be removed by an exact form. The next best thing we can do instead is to determine φ by the requirement that $\zeta_\varphi = \zeta - \mathrm{d}\varphi$ only has simple poles at Q, Q'. Then we define the pairing [P5]

$$\left\langle \hat{\beta}, \zeta \right\rangle := \int_{\hat{\beta}} (\zeta - \mathrm{d}\varphi) = \int_{\hat{\beta}} \zeta_\varphi \ , \tag{6.13}$$

which diverges only logarithmically. To regulate this divergence we introduce a cut-off as before, i.e. we take $\hat{\beta}$ to run from Λ_0' to Λ_0. We will have more to say about this logarithmic divergence below. So although ζ_φ is not a representative of a class in $H^1(\Sigma, \{Q, Q'\})$ it is as close as we can get.

We now want to determine φ explicitly. To keep track of the poles and zeros of the various terms it is useful to apply the theory of divisors, as explained in appendix B.2 and e.g. in [55]. The divisors of various functions and forms on Σ have already been

explained in detail in section 4.1. Consider now $\varphi_k := \frac{x^k}{y}$ with $\mathrm{d}\varphi_k = \frac{kx^{k-1}\mathrm{d}x}{y} - \frac{x^k y^{2\prime}\mathrm{d}x}{2y^3}$. For x close to Q or Q' the leading term of this expression is $\pm(k-\hat{g}-1)x^{k-\hat{g}-2}\mathrm{d}x$. This has no pole at Q, Q' for $k \leq \hat{g}$, and for $k = \hat{g}+1$ the coefficient vanishes, so that we do not get simple poles at Q, Q'. This is as expected as $\mathrm{d}\varphi_k$ is exact and cannot contain poles of first order. For $k \geq \hat{g}+2 = n+1$ the leading term has a pole of order $k-\hat{g}$ and so $\mathrm{d}\varphi_k$ contains poles of order $k-\hat{g}, k-\hat{g}-1, \ldots 2$ at Q, Q'. Note also that at $P_1, \ldots P_{2\hat{g}+2}$ one has double poles for all k (unless a zero of y occurs at $x=0$). Next, we set

$$\varphi = \frac{\mathcal{P}}{y}, \tag{6.14}$$

with $\mathcal{P}$ a polynomial of order $2\hat{g}+3$. Then $\mathrm{d}\varphi$ has poles of order $\hat{g}+3, \hat{g}+2, \ldots 2$ at Q, Q', and double poles at the zeros of y (unless a zero of $\mathcal{P}_k$ coincides with one of the zeros of y). From the previous discussion it is clear that we can choose the coefficients in $\mathcal{P}$ such that $\zeta_\varphi = \zeta - \mathrm{d}\varphi$ only has a simple pole at Q, Q' and double poles at $P_1, P_2, \ldots P_{2\hat{g}+2}$. Actually, the coefficients of the monomials x^k in $\mathcal{P}$ with $k \leq \hat{g}$ are not fixed by this requirement. Only the $\hat{g}+2$ highest coefficients will be determined, in agreement with the fact that we cancel the $\hat{g}+2$ poles of order $\hat{g}+3, \ldots 2$.

It remains to determine the polynomial $\mathcal{P}$ explicitly. The part of ζ contributing to the poles of order ≥ 2 at Q, Q' is easily seen to be $\pm W'(x)\mathrm{d}x$ and we obtain the condition

$$W'(x) - \left(\frac{\mathcal{P}(x)}{\sqrt{W'(x)^2 + f(x)}}\right)' = o\left(\frac{1}{x^2}\right) . \tag{6.15}$$

Integrating this equation, multiplying by the square root and developing the square root leads to

$$W(x)W'(x) - \frac{2t}{n+1}x^n - \mathcal{P}(x) = cx^n + o\left(x^{n-1}\right) , \tag{6.16}$$

where c is an integration constant. We read off [P5]

$$\varphi(x) = \frac{W(x)W'(x) - \left(\frac{2t}{n+1} + c\right)x^n + o\left(x^{n-1}\right)}{y} , \tag{6.17}$$

and in particular, for x close to infinity on the upper or lower sheet,

$$\varphi(x) \sim \pm\left[W(x) - c + o\left(\frac{1}{x}\right)\right] . \tag{6.18}$$

The arbitrariness in the choice of c has to do with the fact that the constant $W(0)$ does not appear in the description of the Riemann surface. In the sequel we will choose $c = 0$, such that the full $W(x)$ appears in (6.18). As is clear from our construction, and is easily verified explicitly, close to Q, Q' one has $\zeta_\varphi \sim \left(\mp\frac{2t}{x} + o\left(\frac{1}{x^2}\right)\right)\mathrm{d}x$.

With this φ we find

$$\int_{\hat{\beta}} \zeta_\varphi = \int_{\hat{\beta}} \zeta - \int_{\hat{\beta}} \mathrm{d}\varphi = \int_{\hat{\beta}} \zeta - \varphi(\Lambda_0) + \varphi(\Lambda_0') = \int_{\hat{\beta}} \zeta - 2\left(W(\Lambda_0) + o\left(\frac{1}{\Lambda_0}\right)\right) . \tag{6.19}$$

Note that, contrary to ζ, ζ_φ has poles at the zeros of y, but these are double poles and it does not matter how the cycle is chosen with respect to the location of these poles (as long as it does not go right through the poles). Note also that we do not need to evaluate the integral of ζ_φ explicitly. Rather one can use the known result (5.85) for the integral of ζ to find from (6.19)

$$\frac{1}{2}\left\langle \hat{\beta}, \zeta \right\rangle = \frac{1}{2}\int_{\hat{\beta}} \zeta_\varphi = \frac{\partial}{\partial t}\mathcal{F}_0(t, \tilde{S}) - t\log\Lambda_0^2 + o\left(\frac{1}{\Lambda_0}\right) \ . \tag{6.20}$$

Let us comment on the independence of the representative of the class $[\zeta] \in H^1(\hat{\Sigma})$. Suppose we had started from $\tilde{\zeta} := \zeta + \mathrm{d}\rho$ instead of ζ. Then determining $\tilde{\varphi}$ by the same requirement that $\tilde{\zeta} - \mathrm{d}\tilde{\varphi}$ only has first order poles at Q and Q' would have led to $\tilde{\varphi} = \varphi + \rho$ (a possible ambiguity related to the integration constant c again has to be fixed). Then obviously

$$\left\langle \hat{\beta}, \tilde{\zeta} \right\rangle = \int_{\hat{\beta}} \tilde{\zeta} - \int_{\partial\hat{\beta}} \tilde{\varphi} = \int_{\hat{\beta}} \zeta - \int_{\partial\hat{\beta}} \varphi = \left\langle \hat{\beta}, \zeta \right\rangle \ , \tag{6.21}$$

and hence our pairing only depends on the cohomology class $[\zeta]$.

Finally, we want to lift the discussion to the local Calabi-Yau manifold. There we define the pairing

$$\left\langle \Gamma_{\hat{\beta}}, \Omega \right\rangle := \int_{\Gamma_{\hat{\beta}}} (\Omega - \mathrm{d}\Phi) = (-i\pi)\int_{\hat{\beta}} (\zeta - \mathrm{d}\varphi) \ , \tag{6.22}$$

where (recall that $c = 0$)

$$\Phi := \frac{W(x)W'(x) - \frac{2t}{n+1}x^n}{W'(x)^2 + f_0(x)} \cdot \frac{\mathrm{d}v \wedge \mathrm{d}w}{2z} \tag{6.23}$$

is such that $\int_{\Gamma_{\hat{\beta}}} \mathrm{d}\Phi = -i\pi \int_{\hat{\beta}} \mathrm{d}\varphi$. Clearly, we have

$$-\frac{1}{2\pi i}\left\langle \Gamma_{\hat{\beta}}, \Omega \right\rangle = \frac{\partial \mathcal{F}_0(t, \tilde{S})}{\partial t} - t\log\Lambda_0^2 + o\left(\frac{1}{\Lambda_0}\right) \ . \tag{6.24}$$

6.1.2 The superpotential and matrix models

Let us now return to the effective superpotential W_{eff} in (6.6). Following [27] and [45] we have for the integrals of G_3 over the cycles Γ_A and Γ_B:

$$N_i = \int_{\Gamma_{A^i}} G_3 \quad , \quad \tau_i := \langle \Gamma_{B_i}, G_3 \rangle \qquad \text{for} \ \ i = 1, \ldots n \ . \tag{6.25}$$

Here we used the fact that the integral over the flux on the deformed space should be the same as the number of branes in the resolved space. It follows for the integrals

over the cycles Γ_α and Γ_β

$$\begin{aligned}\tilde{N}_i := \int_{\Gamma_{\alpha^i}} G_3 = \sum_{j=1}^{i} N_j \quad &, \quad \tilde{\tau}_i := \int_{\Gamma_{\beta_i}} G_3 = \tau_i - \tau_{i+1} \quad \text{for} \;\; i = 1, \dots n-1 \;, \\ N = \sum_{i=1}^{n} N_i = \int_{\Gamma_{\hat{\alpha}}} G_3 \quad &, \quad \tilde{\tau}_0 := \left\langle \Gamma_{\hat{\beta}}, G_3 \right\rangle = \tau_n \;. \end{aligned} \tag{6.26}$$

For the non-compact cycles, instead of the usual integrals, we use the pairings of the previous section. On the Calabi-Yau, the pairings are to be understood e.g. as $\tau_i = -i\pi \langle B_i, h \rangle$, where $\int_{S^2} G_3 = -2\pi i h$ and S^2 is the sphere in the fibre of $\Gamma_{B_i} \to B_i$. Note that this implies that the τ_i as well as $\tilde{\tau}_0$ have (at most) a logarithmic divergence, whereas the $\tilde{\tau}_i$ are finite. We propose [P5] that the superpotential should be defined as

$$\begin{aligned} &W_{eff}(t, \tilde{S}_i) = \\ &= \frac{1}{2\pi i} \sum_{i=1}^{n-1} \left(\int_{\Gamma_{\alpha^i}} G_3 \int_{\Gamma_{\beta_i}} \Omega - \int_{\Gamma_{\beta_i}} G_3 \int_{\Gamma_{\alpha^i}} \Omega \right) + \frac{1}{2\pi i} \left(\int_{\Gamma_{\hat{\alpha}}} G_3 \cdot \left\langle \Gamma_{\hat{\beta}}, \Omega \right\rangle - \left\langle \Gamma_{\hat{\beta}}, G_3 \right\rangle \int_{\Gamma_{\hat{\alpha}}} \Omega \right) \\ &= -\frac{1}{2} \sum_{i=1}^{n-1} \left(\tilde{N}_i \int_{\beta_i} \zeta - \tilde{\tau}_i \int_{\alpha^i} \zeta \right) - \frac{1}{2} \left(N \left\langle \hat{\beta}, \zeta \right\rangle - \tilde{\tau}_0 \int_{\hat{\alpha}} \zeta \right) \;. \end{aligned} \tag{6.27}$$

This formula is very similar to the one advocated for example in [99], but now the pairing (6.13) is to be used. Note that Eq. (6.27) is invariant under symplectic transformations on the basis of (relative) three-cycles on the local Calabi-Yau manifold, resp. (relative) one-cycles on the Riemann surface, provided one uses the pairing (6.13) for every relative cycle. These include $\alpha^i \to \beta_i,\ \hat{\alpha} \to \hat{\beta},\ \beta_i \to -\alpha^i,\ \hat{\beta} \to -\hat{\alpha}$, which acts as electric-magnetic duality.

It is quite interesting to note what happens to our formula for the superpotential in the classical limit, in which we have $\tilde{S}_i = t = 0$, i.e. $f_0(x) \equiv 0$. In this case we have $\zeta = \mathrm{d}W$ and we find

$$\begin{aligned} W_{eff} &= \sum_{i=1}^{n-1} \tilde{N}_i \int_{\mu_i}^{\mu_{i+1}} \mathrm{d}W + \frac{N}{2} \left(2 \int_{\mu_n}^{\Lambda_0} \mathrm{d}W - \int_{\Lambda_0'}^{\Lambda_0} \mathrm{d} \left(\frac{W(x)W'(x) + o\left(x^{n-1}\right)}{y} \right) \right) \\ &= \sum_{i=1}^{n} N_i W(\mu_i) \;, \end{aligned} \tag{6.28}$$

where μ_i are the critical points of W. But this is nothing but the value of the tree-level superpotential in the vacuum in which $U(N)$ is broken to $U(N_i)$.

By now it should be clear that the matrix model analysis was indeed very useful to determine physical quantities. Not only do we have a precise understanding of the divergences, but we can now also rewrite the effective superpotential in terms of the

matrix model free energy. Using the special geometry relations (5.80), (5.81) for the standard cycles and (5.82), (6.20) for the relative cycles, we obtain

$$\begin{aligned} W_{eff} &= -\sum_{i=1}^{n-1} \tilde{N}_i \frac{\partial}{\partial \tilde{S}_i} \mathcal{F}_0(t, \tilde{S}_1, \ldots, \tilde{S}_{n-1}) + 2\pi i \sum_{i=1}^{n-1} \tilde{\tau}_i \tilde{S}_i \\ & \quad -N \frac{\partial}{\partial t} \mathcal{F}_0(t, \tilde{S}_1, \ldots, \tilde{S}_{n-1}) + \left(N \log \Lambda_0^2 + 2\pi i \tilde{\tau}_0\right) t + o\left(\frac{1}{\Lambda_0}\right) . \end{aligned} \tag{6.29}$$

The limit $\Lambda_0 \to \infty$ can now be taken provided $N \log \Lambda_0^2 + 2\pi i \tilde{\tau}_0$ is finite. In other words, $2\pi i \tilde{\tau}_0$ itself has to contain a term $-N \log \Lambda_0^2$ which cancels against the first one. This is of course quite reasonable, since $\tilde{\tau}_0$ is defined via the pairing $\left\langle \hat{\beta}, h \right\rangle$, which is expected to contain a logarithmic divergence. Note that $\tilde{\tau}_0$ is the only flux number in (6.29) that depends on Λ_0. This logarithmic dependence on some scale parameter is of course familiar from quantum field theory, where we know that the coupling constants depend logarithmically on some energy scale. It is then very natural to identify [27] the flux number $\tilde{\tau}_0$ with the gauge theory bare coupling as

$$\tilde{\tau}_0 = \frac{4\pi i}{g_0^2} + \frac{\Theta}{2\pi} . \tag{6.30}$$

In order to see this in more detail note that our gauge theory with a chiral superfield in the adjoint has a β-function $\beta(g) = -\frac{2N}{16\pi^2} g^3$. This leads to $\frac{1}{g^2(\mu)} = \frac{2N}{8\pi^2} \log\left(\frac{\mu}{|\Lambda|}\right)$. If $\tilde{\Lambda}_0$ is the cut-off of the gauge theory we have $\frac{1}{g_0^2} \equiv \frac{1}{g^2(\tilde{\Lambda}_0)} = \frac{1}{g^2(\mu)} + \frac{2N}{8\pi^2} \log\left(\frac{\tilde{\Lambda}_0}{\mu}\right)$. We now have to identify the gauge theory cut-off $\tilde{\Lambda}_0$ with the cut-off Λ_0 on the Riemann surface as

$$\Lambda_0 = \tilde{\Lambda}_0 \tag{6.31}$$

to obtain a finite effective superpotential. Indeed, then one gets

$$N \log \Lambda_0^2 + 2\pi i \tilde{\tau}_0 = 2\pi i \tilde{\tau}(\mu) + 2N \log \mu = 2N \log \Lambda , \tag{6.32}$$

with finite $\Lambda = |\Lambda| e^{i\Theta/2N}$ and $\tilde{\tau}(\mu)$ as in (6.30), but now with $g(\mu)$ instead of g_0. Note that Λ is the dynamical scale of the gauge theory with dimension 1, which has nothing to do with the cut-off Λ_0.

Eq. (6.29) can be brought into the form of [45] if we use the coordinates $\bar{S}_i$, as defined in (5.92) and such that $\bar{S}_i = \frac{1}{4\pi i} \int_{A^i} \zeta$ for all $i = 1, \ldots n$. We get

$$W_{eff} = -\sum_{i=1}^{n} N_i \frac{\partial}{\partial \bar{S}_i} \mathcal{F}_0(\bar{S}) + \sum_{i=1}^{n-1} \bar{S}_i \left(2\pi i \sum_{j=i}^{n-1} \tilde{\tau}_j + \log \Lambda^{2N} \right) + \bar{S}_n \log \Lambda^{2N} . \tag{6.33}$$

In order to compare to [45] we have to identify $\frac{\partial \mathcal{F}_0(S)}{\partial \bar{S}_i}$ with $\frac{\partial \mathcal{F}_0^p(S)}{\partial \bar{S}_i} + \bar{S}_i \log \bar{S}_i + \ldots$, where $\mathcal{F}_0^p$ is the perturbative part of the free energy of the matrix model. Indeed, it was argued in [45] that the $\bar{S}_i \log \bar{S}_i$ terms come from the measure and are contained

in the non-perturbative part $\frac{\partial \mathcal{F}_0^{np}}{\partial S_i}$. In fact, the presence of these terms in $\frac{\partial \mathcal{F}_0^{np}}{\partial S_i}$ can easily be proven by monodromy arguments [27]. Alternatively, the presence of the $S_i \log S_i$-terms in $\mathcal{F}_0$ can be proven by computing $\mathcal{F}_0$ exactly in the planar limit. We will discuss some explicit examples below.

We could have chosen $\hat{\beta}$ to run from a point $\Lambda_0' = |\Lambda_0| e^{i\theta/2}$ on the lower sheet to a point $\Lambda_0 = |\Lambda_0| e^{i\theta/2}$ on the upper sheet. Then one would have obtained an additional term $-it\theta$, on the right-hand side of (6.20), which would have led to

$$\Theta \to \Theta + N\theta \tag{6.34}$$

in (6.33).

Note that (6.33) has dramatic consequences. In particular, after inserting $\mathcal{F}_0 = \mathcal{F}_0^p + \mathcal{F}_0^{np}$ we see that the effective superpotential, which upon extremisation gives a non-perturbative quantity in the gauge theory can be calculated from a perturbative expansion in a corresponding matrix model. To be more precise, $\mathcal{F}_0^p$ can be calculated by expanding the matrix model around a vacuum in which the filling fractions ν_i^* are fixed in such a way that the pattern of the gauge group $\prod_{i=1}^n U(N_i)$ is reproduced. This means that whenever $N_i = 0$ we choose $\nu_i^* = 0$ and whenever $N_i \neq 0$ we have $\nu_i^* \neq 0$. Otherwise the ν_i^* can be chosen arbitrarily (and are in particular independent of the N_i.) $\mathcal{F}_0^p$ is then given by the sum of all planar vacuum amplitudes.

As was shown in [45] this can be generalised to more complicated $\mathcal{N} = 1$ theories with different gauge groups and field contents.

6.2 Example: Superstrings on the conifold

Next we want to illustrate our general discussion by looking at the simplest example, i.e. $n = 1$ and $W = \frac{x^2}{2}$. This means we study type IIB string theory on the resolved conifold. Wrapping N D5-branes around the single $\mathbb{CP}^1$ generates $\mathcal{N} = 1$ Super-Yang-Mills theory with gauge group $U(N)$ in four dimensions. The corresponding low energy effective superpotential is well-known to be the Veneziano-Yankielowicz superpotential. As a first test of the claim of Cachazo-Intriligator Vafa we want to reproduce this superpotential.

According to our recipe we have to take the space through a geometric transition and evaluate Eq. (6.27) on the deformed space. From our discussion in section 4.2 we know that this is given by the deformed conifold,

$$x^2 + v^2 + w^2 + z^2 - \mu = 0 \ , \tag{6.35}$$

where we took $f(x) = -\mu = -4t$, $\mu \in \mathbb{R}^+$. The integrals of ζ on the corresponding Riemann surface have already been calculated in (5.109) and (5.110). Obviously one has $\zeta = -2t\frac{\mathrm{d}x}{y} + \mathrm{d}\left(\frac{xy}{2}\right)$, which would correspond to $\varphi = \frac{xy}{2}$. Comparing with (6.17) this would yield $c = t$. The choice $c = 0$ instead leads to $\varphi = \frac{xy}{2} + t\frac{x}{y}$ and $\zeta = -2t\frac{\mathrm{d}x}{y} + 4t^2\frac{\mathrm{d}x}{y^3} + \mathrm{d}\varphi$. The first term has a pole at infinity and leads to the logarithmic

divergence, while the second term has no pole at infinity but second order poles at $\pm 2\sqrt{t}$. One has

$$2\varphi(\Lambda_0) = \Lambda_0\sqrt{\Lambda_0^2 - 4t} + 2t\frac{\Lambda_0}{\sqrt{\Lambda_0^2 - 4t}} \ . \tag{6.36}$$

and

$$\begin{aligned}\frac{1}{2}\left\langle \hat{\beta}, \zeta \right\rangle &= t\log\left(\frac{4t}{\Lambda_0^2}\right) - 2t\log\left(1 + \sqrt{1 - \frac{4t}{\Lambda_0^2}}\right) - t\frac{1}{\sqrt{1 - \frac{4t}{\Lambda_0^2}}} \\ &= \frac{\partial \mathcal{F}_0(t)}{\partial t} - t\log\Lambda_0^2 + o\left(\frac{1}{\Lambda_0^2}\right) \ ,\end{aligned} \tag{6.37}$$

where we used (5.110) and the explicit form of $\mathcal{F}_0(t)$, (5.107). Finally, in the present case, Eq. (6.27) for the superpotential only contains the relative cycles,

$$W_{eff} = -\frac{N}{2}\left\langle \hat{\beta}, \zeta \right\rangle + \frac{\tilde{\tau}_0}{2}\int_{\hat{\alpha}} \zeta \tag{6.38}$$

or

$$W_{eff} = -N\left(t\log t - t - t\log\Lambda_0^2\right) + 2\pi i\tilde{\tau}_0 t + o\left(\frac{1}{\Lambda_0^2}\right) \ . \tag{6.39}$$

For $U(N)$ super Yang-Mills theory the β-function reads $\beta(g) = -\frac{3N}{16\pi^2}g^3$ and one has to use the identification $\Lambda_0^2 = \tilde{\Lambda}_0^3$ between the geometric cut-off Λ_0 and the gauge theory cut-off $\tilde{\Lambda}_0$. Then $N\log\Lambda_0^2 + 2\pi i\tau_0 = 3N\log|\Lambda| + i\Theta = 3N\log\Lambda$. Therefore, sending the cut-off Λ_0 to infinity, and using $\bar{S} = t$, we indeed find the Veneziano-Yankielowicz superpotential,

$$W_{eff} = \bar{S}\log\left(\frac{\Lambda^{3N}}{\bar{S}^N}\right) + \bar{S}N \ . \tag{6.40}$$

6.3 Example: Superstrings on local Calabi-Yau manifolds

After having studied superstrings on the conifold we now want to extend these considerations to the more complicated local Calabi-Yau spaces X_{res}. In other words, we want to study the low energy effective superpotential of an $\mathcal{N} = 1$ $U(N)$ gauge theory coupled to a chiral superfield Φ in the adjoint, with tree-level superpotential $W(\Phi)$. The general structure of this gauge theory has been analysed using field theory methods in [27], see also [28], [29]. There the authors made use of the fact that the gauge theory can be understood as an $\mathcal{N} = 2$ theory which has been broken to $\mathcal{N} = 1$ by switching on the tree-level superpotential. Therefore, one can apply the exact results of Seiberg and Witten [122] on $\mathcal{N} = 2$ theories to extract some information about the $\mathcal{N} = 1$ theory. In particular, the form of the low energy effective superpotential W_{eff}

can be deduced [27]. Furthermore, by studying the monodromy properties of the geometric integrals, the general structure of the Gukov-Vafa-Witten superpotential was determined, and the structures of the two superpotentials were found to agree, which provides strong evidence for the conjecture (6.8). For the case of the cubic superpotential the geometric integrals were evaluated approximately, and the result agreed with the field theory calculations.

Unfortunately, even for the simplest case of a cubic superpotential the calculations are in general quite involved. Therefore, we are going to consider a rather special case in this section. We choose $n = 2$, i.e. the tree-level superpotential W is cubic, and we have 2 $\mathbb{CP}^1$s in X_{res}, corresponding to the small resolution of the two singularities in $W'(x)^2 + v^2 + w^2 + z^2 = 0$. In order for many of the calculations to be feasible, we choose the specific vacuum in which the gauge group remains unbroken. In other words, we wrap all the N physical D-branes around one of the two $\mathbb{CP}^1$s, e.g. $N_1 = 0$ and $N_2 = N$. Therefore, strings can end on only one $\mathbb{CP}^1$. We saw already that the pattern of the breaking of the gauge group is mirrored in the filling fractions ν_i^* of the matrix model. In our case we must have $\nu_1^* = 0$ and $\nu_2^* = 1$.[5] But from $\frac{1}{4\pi^2}\int_{\Gamma_{A^i}} \Omega = \bar{S}_i = t\nu_i^*$ we learn that the corresponding deformed geometry X_{def} contains a cycle Γ_{A^1} of vanishing size, together with the finite Γ_{A^2}. This situation is captured by a hyperelliptic Riemann surface, where the two complex planes are connected by one cut and one point, see Fig. 6.1 The situation in which one cut collapsed to zero size is described mathematically by

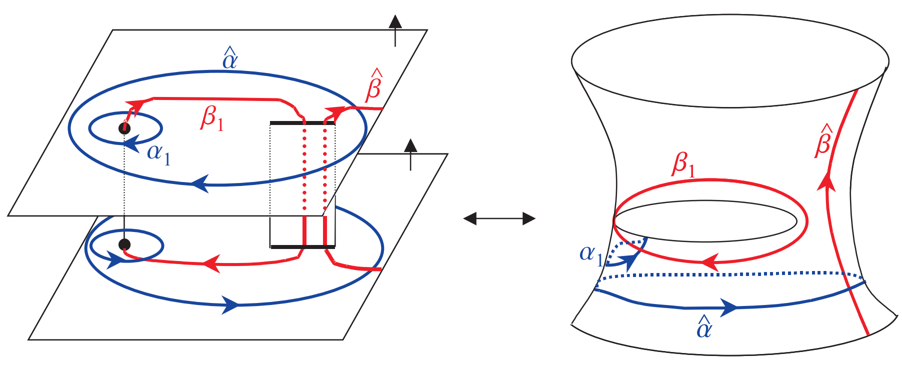

Figure 6.1: The Riemann surface for a cubic tree level potential, and f_0 such that one of the two cuts collapses to a point.

a double zero of the polynomial defining the Riemann surface. Therefore, the vacuum with unbroken gauge group leads to the surface

$$y^2 = g^2(x-a)^2(x-b)(x+b) = \left(gx^2 - agx - \frac{b^2 g}{2}\right)^2 + ab^2g^2x - a^2b^2g^2 - \frac{b^4g^2}{4} \; . \quad (6.41)$$

[5]This can also be understood by looking at the topological string. As we will see in the next chapter, the B-type topological string with $\hat{N}$ topological branes wrapping the two $\mathbb{CP}^1$s calculates terms in the four-dimensional effective action of the superstring theory. If there are no D5-branes wrapping $\mathbb{CP}^1$ the related topological strings will also not be allowed to end on this $\mathbb{CP}^1$, i.e. there are no topological branes wrapping it. This amounts to $\hat{N}_1 = 0$ and $\hat{N}_2 = \hat{N}$. In the next chapter we will see that the number of topological branes and the filling fractions are related as $\nu_i^* = \hat{N}_i/\hat{N}$.

Without loss of generality we took the cut to be symmetric around zero. We want both the cut and the double zero to lie on the real axis, i.e. we take g real and positive and $a, b \in \mathbb{R}$. From (6.41) we can read off the superpotential,

$$W(x) = \frac{g}{3}x^3 - \frac{ag}{2}x^2 - \frac{b^2 g}{2}x \; , \tag{6.42}$$

where we chose $W(0) = 0$. Note that, in contrast to most of the discussion in chapter 5, the leading coefficient of W is $\frac{g}{3}$ and not $\frac{1}{3}$, because we want to keep track of the coupling g. This implies (c.f. Eq. (5.51)) that the leading coefficient of f_0 is $-4tg$, and therefore

$$t = -\frac{ab^2 g}{4} \; . \tag{6.43}$$

Since t is taken to be real and positive, a must be negative. Furthermore, we define the positive $m := -ag$. Then $W(x) = \frac{g}{3}x^3 + \frac{m}{2}x^2 - \frac{2tg}{m}x$. Note that we have in this special setup $\bar{S}_1 = \tilde{S}_1 = \frac{1}{4\pi^2}\int_{\Gamma_{\alpha 1}} \Omega = 0$ and $\bar{S}_2 = \frac{1}{4\pi^2}\int_{\Gamma_{\hat{\alpha}}} \Omega = \frac{-i}{4\pi}\int_{\hat{\alpha}} \zeta = \frac{-i}{4\pi}\int_{\hat{\alpha}} y(x)\mathrm{d}x = t$. Using (5.47) and (5.52) our curve (6.41) leads to the spectral density

$$\rho_0(x) = \frac{g}{2\pi t}(x-a)\sqrt{b^2 - x^2} \quad \text{for } x \in [-b, b] \tag{6.44}$$

and zero otherwise. The simplest way to calculate the planar free energy is by making use of the homogeneity relation (5.86). We find

$$\begin{aligned} \mathcal{F}_0(t) &= \frac{t}{2}\frac{\partial \mathcal{F}_0(t)}{\partial t} - \frac{t}{2}\int \mathrm{d}s \; \rho_0(s) W(s) \\ &= \frac{t}{2}\lim_{\Lambda_0 \to \infty}\left(\frac{1}{2}\int_{\hat{\beta}} \zeta - W(\Lambda_0) + t\log \Lambda_0^2\right) - \frac{t}{2}\int_{-b}^{b} \mathrm{d}s \; \rho_0(s) W(s) \; . \end{aligned} \tag{6.45}$$

Here we used that the cut between $-b$ and b lies on the real axis, where we can take $\lambda(s) = s$. In the second line we made use of (5.85). The integrals can be evaluated without much effort, and the result is

$$\mathcal{F}_0(t) = \frac{t^2}{2}\log\frac{t}{m} - \frac{3}{4}t^2 + \frac{2}{3}\frac{g^2}{m^3}t^3 \; . \tag{6.46}$$

This result is very interesting, since it contains the planar free energy of the conifold (5.107), which captures the non-perturbative contributions, together with a perturbative term $\frac{2}{3}\frac{t^3}{m^3}g^2$. Note that this is precisely the term calculated from matrix model perturbation theory in Eq. (5.23). (Recall that in our special case $\mathcal{F}_0(t) = -F_0(t)$, c.f. Eq. (5.62), which accounts for the minus sign.) On the other hand, it seems rather puzzling that there are no $o(g^4)$-terms in the free energy. In particular, (5.23) contains a term $-\frac{8}{3}\frac{t^4}{m^6}g^4$, coming from fatgraph diagrams containing four vertices. However, there we had a slightly different potential, namely $W(x) = \frac{g}{3}x^3 + \frac{m}{2}x^2$, i.e. there was no linear term. In our case this linear term is present, leading to tadpoles in the Feynman diagrams. Each tadpole comes with a factor $\frac{2tg}{mg_s}$. Then, next to the planar

Figure 6.2: The linear term in W leads to tadpoles in the Feynman diagrams, each tadpole coming with $\frac{2tg}{mg_s}$, which contains a factor of g. At order g^2 only two diagrams contribute.

diagrams of Fig. 5.4, there are two more diagrams that contribute to F_0 at order g^2, see Fig. 6.2. However, their contribution cancels, because of different symmetry factors. This is why, in the presence of tadpoles, we still reproduce the result of (5.22) at order g^2. Furthermore, the explicit expression for $\mathcal{F}_0(t)$ tells us that starting from order g^4 (as in the case without tadpoles there are no diagrams that lead to an odd power of g) the contributions coming from tadpole diagrams cancel the contributions of the diagrams without tadpoles.

It is quite interesting to look at this from a slightly different perspective. We again start from the potential $W(x) = \frac{g}{3}x^3 + \frac{m}{2}x^2 - \frac{2tg}{m}x$ and perform the shift $x = y + \alpha := y + \frac{m}{2g}\left(\sqrt{1+\frac{8tg^2}{m^3}} - 1\right)$. Then we have $W(x) = \tilde{W}(y)$ with

$$\tilde{W}(y) = \frac{g}{3}y^3 + \frac{m}{2}\Delta y^2 + \tilde{W}_0 \ , \tag{6.47}$$

where $\Delta := \sqrt{1+\frac{8tg^2}{m^3}}$ and $W_0 = \frac{m^3}{12g^2}\left(-\frac{1}{2} + \frac{3}{2}\Delta^2 - \Delta^3\right)$. The shift of x was chosen in such a way that $\tilde{W}(y)$ does not contain a term linear in y.

Next consider what happens at the level of the partition function. We have

$$\begin{aligned} Z &= \int DMe^{-\frac{1}{g_s}\operatorname{tr} W(M)} = \int D\tilde{M}e^{-\frac{1}{g_s}\operatorname{tr}\tilde{W}(\tilde{M})} = e^{-\frac{N\tilde{W}_0}{g_s}} \int D\tilde{M}e^{-\frac{1}{g_s}\operatorname{tr}\left(\frac{g}{3}\tilde{M}^3 + \frac{m\Delta}{2}\tilde{M}^2\right)} \\ &= \exp\left(-\frac{t\tilde{W}_0}{g_s^2} - \frac{t^2}{2g_s^2}\log\Delta\right)\int D\hat{M}\exp\left(-\frac{1}{g_s}\operatorname{tr}\left(\frac{g\Delta^{-3/2}}{3}\hat{M}^3 + \frac{m}{2}\hat{M}^2\right)\right) \end{aligned} \tag{6.48}$$

This shows that partition functions with different coefficients in the defining polynomial potential are related as

$$Z\left(g, m, -\frac{2tg}{m}\right) = e^{-\frac{1}{g_s^2}\left(t\tilde{W}_0 + \frac{t^2}{2}\log\Delta\right)} Z\left(g\Delta^{-3/2}, m, 0\right) \ . \tag{6.49}$$

In terms of the planar free energy this is

$$\begin{aligned} \mathcal{F}_0(g\Delta^{-3/2}, m, 0) &= \mathcal{F}_0\left(g, m, -\frac{2tg}{m}\right) + t\tilde{W}_0(g,m,t) + \frac{t^2}{2}\log\Delta(g,m,t) \\ &= \mathcal{F}_0\left(g, m, -\frac{2tg}{m}\right) + t^2\left(\frac{-1+3\Delta^2-2\Delta^3}{3(\Delta^2-1)} + \frac{1}{2}\log\Delta\right) \end{aligned} \tag{6.50}$$

The first term on the right-hand side of this equation is the one we calculated in (6.46). Therefore,

$$\begin{aligned}\mathcal{F}_0(g\Delta^{-3/2}, m, 0) &= \frac{t^2}{2}\log\frac{t}{m} - \frac{3}{4}t^2 + \frac{t^2}{12}(\Delta^2-1) + \frac{t^3}{3}\left(\frac{1-3\Delta^2+2\Delta^3}{1-\Delta^2} + \frac{3}{2}\log\Delta\right) \\ &= \frac{t^2}{2}\log\frac{t}{m} - \frac{3}{4}t^2 + \mathcal{F}_0^p\ . \end{aligned} \tag{6.51}$$

Note that this expression is exact, and it contains both the perturbative and the non-perturbative part of $\mathcal{F}_0$ for a potential that only contains a cubic and a quadratic term. In other words, expanding this expression in terms of $\tilde{g}^2 := g^2\Delta^{-3}$ we reproduce the contributions of all the planar fatgraph Feynman diagrams. This can be done by using $\Delta^2 = 1 + \frac{8t}{m^3}\tilde{g}^2\Delta^3$ to express Δ in terms of $\tilde{g}^2$. The result is

$$\begin{aligned}\mathcal{F}_0(\tilde{g}, m, 0) &= t^2\left\{\frac{1}{2}\log\frac{t}{m} - \frac{3}{4} + \frac{2}{3}\frac{t\tilde{g}^2}{m^3} + \frac{8}{3}\left(\frac{t\tilde{g}^2}{m^3}\right)^2 + \frac{56}{3}\left(\frac{t\tilde{g}^2}{m^3}\right)^3 + \frac{512}{3}\left(\frac{t\tilde{g}^2}{m^3}\right)^4\right. \\ &\quad \left. + \frac{9152}{5}\left(\frac{t\tilde{g}^2}{m^3}\right)^5 + \ldots\right\} \\ &= t^2\left\{\frac{1}{2}\log\frac{t}{m} - \frac{3}{4} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{(8\frac{t\tilde{g}^2}{m^3})^k}{(k+2)!}\frac{\Gamma(3k/2)}{\Gamma(k/2+1)}\right\}\ . \end{aligned} \tag{6.52}$$

This is precisely the expression found in [25].

Coming now back to our original form of the potential $W(x) = \frac{g}{3}x^3 + \frac{m}{2}x^2 - \frac{2tg}{m}x$, it remains to write down the effective superpotential. For that purpose we write $S := t$. From (6.46) we find

$$\frac{\partial\mathcal{F}_0(S)}{\partial S} = S\log S - S\log m - S + 2\frac{g^2}{m^3}S^2\ . \tag{6.53}$$

As promised below (6.33), this contains a term $S\log S$. Plugging this into (6.33) gives

$$W_{eff}(S) = NS + S\log\left(\frac{\Lambda^{2N}m^N}{S^N}\right) - 2N\frac{g^2}{m^3}S^2\ . \tag{6.54}$$

Note that we also have a term $S\log m$ in the derivative of $\mathcal{F}_0$. This is interesting for various reasons. Firstly, so far we did not keep the dimensions of the various fields in our theory. The argument of the logarithm in our final result should, however, be dimensionless and since m and Λ have dimension one, and S has dimension three this is indeed the case. Furthermore, one can use the threshold matching condition [27] $\Lambda_L^{3N} = \Lambda^{2N}m^N$ in order to relate our result to the dynamical scale Λ_L of the low-energy pure $\mathcal{N}=1$ super Yang-Mills theory with gauge group $U(N)$. Similar threshold matching conditions also hold for the more general case, see [27] for a discussion.

Chapter 7

B-Type Topological Strings and Matrix Models

In the last chapters we learned how the holomorphic matrix model can be used to calculate the integrals of the holomorphic (3, 0)-form Ω over three-cycles in X_{def}. These integrals in turn are the central building blocks of the effective superpotential of our $U(N)$ gauge theory. The starting point of the argument was the fact that the Riemann surface (4.76), that appears when calculating the integrals, and the one of (5.53), that arises in the 't Hooft limit of the matrix model, are actually the same. However, so far we have not explained the reason *why* these two surfaces agree. What motivated us to study the holomorphic matrix model in the first place? In this final chapter we want to fill this gap and show that the holomorphic matrix model is actually nothing but the string field theory of the open B-type topological string on X_{res}. We will not be able to present a self-contained discussion of this fact, since many of the theories involved are quite complicated and it would take us too far to explain them in detail. The goal of this chapter is rather to familiarise the reader with the central ideas and the main line of argument, without spelling out the mathematics.

Some elementary background material on topological strings is given in appendix D, see [133] and [81] for more details. A recent review of string field theory appeared in [119]. Central for the discussion are the results of the classic article [148], and the holomorphic matrix model first occurred in [43]. A recent review of the entire setup can be found in [104].

The open B-type topological string and its string field theory

It has been known for a long time that in the case of Calabi-Yau compactifications of type II string theory certain terms of the four-dimensional effective action can be calculated by studying topological string theory on the Calabi-Yau manifold [20]. To be specific, consider the case of type IIB string theory on a compact Calabi-Yau manifold X. Then there are terms in the four-dimensional effective action, which (when formulated in $\mathcal{N} = 2$ superspace language) have the form $\int \mathrm{d}^4x\mathrm{d}^4\theta\ F_{\hat{g}}(X^I)(\mathcal{W}^2)^{\hat{g}}$, where X^I are the $\mathcal{N} = 2$ vector multiplets and $\mathcal{W}^2 := \mathcal{W}_{\alpha\beta}\mathcal{W}^{\alpha\beta}$ is built from the chiral $\mathcal{N} = 2$ Weyl multiplet $\mathcal{W}_{\alpha\beta}$, which contains the graviphoton. The function $F_{\hat{g}}$ now turns out

[20] to be nothing but the free energy of the B-type topological string with target space X at genus $\hat{g}$. For example, it is well known that the prepotential governing the structure of vector multiplets in an $\mathcal{N} = 2$ supersymmetric theory is nothing but the genus zero free energy of the topological string on the Calabi-Yau manifold.

Our setup is slightly more complicated, since we are interested in type IIB theory on $M_4 \times X_{res}$, with additional D-branes wrapping the resolved two-cycles. The D-branes lead to an open string sector, and therefore we expect that we have to study the open B-type topological string on X_{res}. In other words, we allow for Riemann surfaces with boundaries as the world-sheet of the topological string. It can be shown that in the B-type topological string, when mapping the world-sheet into the target space, the boundaries have to be mapped to holomorphic cycles in the target space. The appropriate boundary conditions are then Dirichlet along these holomorphic cycles and Neumann in the remaining directions. These boundary conditions amount to introducing "topological branes", which wrap the cycles. See [148], [103], [104] for a discussion of open topological strings and more references. Since the various $\mathbb{CP}^1$s in X_{res} around which the physical D-branes are wrapped are all holomorphic (a fact that we have not proven) we expect that the relevant topological theory is the open B-type topological string with topological branes around the various $\mathbb{CP}^1$s. An obvious question to ask is then whether this topological theory calculates terms in the four-dimensional theory.

As to answer this question we have to take a little detour, and note that the open B-type topological string can actually be described [148] in terms of a cubic string field theory, first introduced in [143]. Usually in string theory the S-matrix is given in terms of a sum over two-dimensional world-sheets embedded in space-time. The corresponding string field theory, on the other hand, is a theory which reproduces this S-matrix from the Feynman rules of a space-time action $S[\Psi]$. Ψ, called the string field, is the fundamental dynamical variable, and it contains infinitely many space-time fields, namely one for each basis state of the standard string Fock space. Writing down the string field theory of a given string theory is a difficult task, and not very much is known about the string field theories of superstrings. However, one does know the string field theory for the open bosonic string [143]. Its action reads

$$S = \frac{1}{g_s} \int \operatorname{tr} \left(\frac{1}{2} \Psi \star Q_{BRST} \Psi + \frac{1}{3} \Psi \star \Psi \star \Psi \right) , \tag{7.1}$$

where g_s is the string coupling. The trace comes from the fact that, once we add Chan-Paton factors, the string field is promoted to a $U(\hat{N})$ matrix of string fields. We will not need the detailed structure of this action, so we just mention that $\star$ is some associative product on the space of string fields and $\int$ is a linear map from the space of string fields to $\mathbb{C}$.[1]

[1] To be more precise, in open string field theory one considers the world-sheet of the string to be an infinite strip parameterised by a spacial coordinate $0 \leq \sigma \leq \pi$ and a time coordinate $-\infty < \tau < \infty$ with flat metric $\mathrm{d}s^2 = \mathrm{d}\sigma^2 + \mathrm{d}\tau^2$. One then considers maps $x : [0, \pi] \to X$ into the target space X. The string field is a functional $\Psi[x(\sigma), \ldots]$, where $\ldots$ stands for the ghost fields $c, \tilde{c}$ in the case of the

The structure of the topological string (see appendix D for some of its properties) is very similar to the bosonic string. In particular there exists a generator Q_B with $Q_B^2 = 0$, the role of the ghost fields is played by some of the particles present in the super multiplets, and the ghost number is replaced by the R-charge. So it seems plausible that an action similar to (7.1) should be the relevant string field theory for the B-type topological string. However, in the case of the open bosonic string the endpoints of the strings are free to move in the entire target space. This situation can be generated in the topological string by introducing topological branes which fill the target space completely. This is reasonable since any Calabi-Yau X is a holomorphic submanifold of itself, so the space filling topological branes do wrap holomorphic cycles. This completes the analogy with the bosonic string, and it was indeed shown in [148] that the string field theory action for the open B-type topological string with space filling topological branes is given by (7.1) with Q_B instead of Q_{BRST}.

In the same article Witten showed that this action does actually simplify enormously. In fact, the string functional is a function of the zero mode of the string, corresponding to the position of the string midpoint, and of oscillator modes. If we decouple all the oscillators the string functional becomes an ordinary function of (target) space-time, the $\star$-product becomes the usual product of functions and the integral becomes the usual integral on target space. In [148] it was shown that this decoupling does indeed take place in the open B-type topological string. This comes from the fact that in the B-model the classical limit is exact (because the Lagrangian is independent of the coupling constant t and therefore one can take $t \to \infty$, which is the classical limit, c.f. the discussion in appendix D). We will not discuss the details of this decoupling but only state the result: the string field theory of the open topological B-model on a Calabi-Yau manifold X with $\hat{N}$ space-time filling topological branes is given by holomorphic Chern-Simons theory on X, with the action

$$S = \frac{1}{2g_s} \int_X \Omega \wedge \,\mathrm{tr}\, \left(\bar{A} \wedge \bar{\partial} \bar{A} + \frac{2}{3} \bar{A} \wedge \bar{A} \wedge \bar{A} \right) \,, \tag{7.4}$$

where $\bar{A}$ is the $(0,1)$-part of a $U(\hat{N})$ gauge connection on the target manifold X.

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bosonic string and for η, θ (c.f. appendix D) in the B-topological string. In [143] Witten defined two operations on the space of functionals, namely integration, as well as an associative, non-commutative star product

$$\int \Psi \quad := \quad \int Dx(\sigma) \prod_{0 \leq \sigma \leq \pi/2} \delta[x(\sigma) - x(\pi - \sigma)] \Psi[x(\sigma)] \,, \tag{7.2}$$

$$\int \Psi_1 \star \ldots \star \Psi_p \quad := \quad \int \prod_{i=1}^{p} Dx_i(\sigma) \prod_{i=1}^{p} \prod_{0 \leq \sigma \leq \pi/2} \delta[x_i(\sigma) - x_{i+1}(\pi - \sigma)] \Psi_i[x_i(\sigma)] \,, \tag{7.3}$$

where $x_{p+1} \equiv x_1$. The integration can be understood as folding the string around its midpoint and gluing the two halves, whereas the star product glues two strings by folding them around their midpoints and gluing the second half of one with the first half of the following. See [117], [148], [104], [119] for more details and references.

Holomorphic matrix models from holomorphic Chern-Simons theory The holomorphic Chern-Simons action (7.4) is the string field theory in the case where we have $\hat{N}$ space-time filling topological branes. However, we are interested in the situation, where the branes only wrap holomorphic two-cycles in X_{res}. Here we will see that in this specific situation the action (7.4) simplifies further.

Before we attack the problem of the open B-type topological string on X_{res} let us first study the slightly simpler target space $\mathcal{O}(-2) \oplus \mathcal{O}(0) \to \mathbb{CP}^1$ (c.f. definition 4.8) with $\hat{N}$ topological branes wrapped around the $\mathbb{CP}^1$. The corresponding string field theory action can be obtained from a dimensional reduction [87], [43], [104], of the action (7.4) on $\mathcal{O}(-2) \oplus \mathcal{O}(0) \to \mathbb{CP}^1$ down to $\mathbb{CP}^1$. Clearly, the original gauge connection $\bar{A}$ leads to a $(0,1)$ gauge field $\bar{a}$ on $\mathbb{CP}^1$, together with two fields in the adjoint, Φ_0 and $\Phi_1^{\bar{y}}$, which are sections of $\mathcal{O}(0)$ and $\mathcal{O}(-2)$ respectively. In other words we take $z, \bar{z}$ to be coordinates on the $\mathbb{CP}^1$, $y, \bar{y}$ are coordinates on the fibre $\mathcal{O}(-2)$ and $x, \bar{x}$ are coordinates on $\mathcal{O}(0)$, and write $\bar{A}(x,\bar{x},y,\bar{y},z,\bar{z}) = \bar{a}(z,\bar{z}) + \Phi_1^{\bar{y}}(z,\bar{z})\mathrm{d}\bar{y} + \Phi_0(z,\bar{z})\mathrm{d}\bar{x}$, where $\bar{a} := \bar{a}_{\bar{z}}\mathrm{d}\bar{z}$. Of course Φ_0 and $\Phi_1^{\bar{y}}$ are in the adjoint representation. The index $\bar{y}$ in $\Phi_1^{\bar{y}}$ reminds us of the fact that $\Phi_1^{\bar{y}}$ transform if we go from one patch of the $\mathbb{CP}^1$ to another, whereas Φ_0 does not. In fact we can build a $(1,0)$-form $\Phi_1 := \Phi_1^{\bar{y}}\mathrm{d}z$. Plugging this form of $\bar{A}$ into (7.4) gives

$$S = \frac{1}{g_s}\int_X \Omega_{xyz}(x,y,z)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}\bar{x} \wedge \mathrm{d}\bar{y} \wedge \,\mathrm{tr}\,\left(\Phi_1 \bar{D}_{\bar{a}} \Phi_0\right) \;, \tag{7.5}$$

where $\bar{D}_{\bar{a}} := \bar{\partial} + [\bar{a},\cdot]$ and $\bar{\partial} := \bar{\partial}_{\bar{z}}\mathrm{d}\bar{z}$. We integrate this over the two line bundles to obtain

$$S = \frac{1}{g_s}\int_{\mathbb{CP}^1} f(z)\,\mathrm{tr}\,\left(\Phi_1 \bar{D}_{\bar{a}} \Phi_0\right) \;. \tag{7.6}$$

But $\mathrm{tr}\,\left(\Phi_1 \bar{D}_{\bar{a}} \Phi_0\right)$ is a (1,1)-form on $\mathbb{CP}^1$ that does not transform if we change the coordinate system. From the invariance of S we deduce that $f(z)$ must be a (holomorphic) function. Since holomorphic functions on $\mathbb{CP}^1$ are constants we have [43]

$$S = \frac{1}{g_s}\int_{\mathbb{P}^1} \mathrm{tr}\,\left(\Phi_1 \bar{D}_{\bar{a}} \Phi_0\right) \;. \tag{7.7}$$

Here we suppressed a constant multiplying the right-hand side.

We are interested in the more general situation in which the target space is X_{res}, which can be understood as a deformation of $\mathcal{O}(-2) \oplus \mathcal{O}(0) \to \mathbb{CP}^1$ by W, see the discussion in section 4.2.2. In particular, we want to study the case in which $\hat{N}_i$ topological branes wrap the i-th $\mathbb{CP}^1$. The string field theory action describing the dynamics of the branes in this situation reads [87], [43], [104]

$$S = \frac{1}{g_s}\int_{\mathbb{P}^1} \mathrm{tr}\,\left(\Phi_1 \bar{D}_{\bar{a}} \Phi_0 + W(\Phi_0)\omega\right) \;, \tag{7.8}$$

where ω is the Kähler class on $\mathbb{CP}^1$ with $\int_{\mathbb{P}^1}\omega = 1$. We will not prove that this is indeed the correct action, but we can at least check whether the equations of motion

lead to the geometric picture of branes wrapping the n $\mathbb{CP}^1$s. As to do so note that the field $\bar{a}$ is just a Lagrange multiplier and it enforces

$$[\Phi_0, \Phi_1] = 0 \ , \tag{7.9}$$

i.e. we can diagonalise Φ_0 and Φ_1 simultaneously. Varying with respect to Φ_1 gives

$$\bar{\partial}\Phi_0 = 0 \ , \tag{7.10}$$

which implies that Φ_0 is constant, as $\mathbb{P}^1$ is compact. Finally, the equation for Φ_0 reads

$$\bar{\partial}\Phi_1 = W'(\Phi_0)\omega \ . \tag{7.11}$$

Integrating both sides over $\mathbb{CP}^1$ gives

$$W'(\Phi_0) = 0 \ , \tag{7.12}$$

for non-singular Φ_1. Plugging this back into (7.11) gives $\bar{\partial}\Phi_1 = 0$. But there are no holomorphic one forms on $\mathbb{CP}^1$, implying $\Phi_1 \equiv 0$. All this tells us that classical vacua are described by $\Phi_1 = 0$ and a diagonal Φ_0, where the entries on the diagonal are constants, located at the critical points of W. But of course, from our discussion of X_{res} we know that these critical points describe the positions of the various $\mathbb{CP}^1$s in X_{res}. Since the eigenvalues of Φ_0 describe the position of the topological branes we are indeed led to our picture of $\hat{N}_i$ topological branes wrapping the i-th $\mathbb{CP}^1$.

After having seen that the classical configurations of our string field theory action do indeed reproduce our geometric setup we now turn back to the action itself. We note that both $\bar{a}$ and Φ_1 appear linearly in (7.8), and hence they can be integrated out. As we have seen, this results in the constraint $\bar{\partial}\Phi_0 = 0$, which means that Φ_0 is a *constant* $\hat{N} \times \hat{N}$ *matrix*,

$$\Phi_0(z) \equiv \Phi = \text{const} \ . \tag{7.13}$$

Now we can plug this solution of the equations of motion back into the action, which then reduces to

$$S = \frac{1}{g_s} \operatorname{tr} W(\Phi). \tag{7.14}$$

But since Φ is a constant $\hat{N} \times \hat{N}$ matrix we find that the string field theory partition function is nothing but a holomorphic matrix model with potential $W(x)$. An alternative derivation of this fact has been given in [104]. It is quite important to note that the number of topological branes $\hat{N}$ is unrelated to the number N of physical D-branes. Indeed, $\hat{N}$ is the size of the matrices in the matrix model and we have seen that interesting information about the physical $U(N)$ gauge theory can be obtained by taking $\hat{N}$ to infinity. This now completes the logic of our reasoning and finally tells us why the holomorphic matrix model can be used in order to extract information about our model.

Open-closed duality

Let us now analyse what the above discussion implies for the relations between the various free energies involved in our setup. Like any gauge theory, the free energy of the matrix model can be expanded in terms of fatgraphs, as discussed in the introduction and in section 5.1.2. Such an expansion leads to quantities $\mathcal{F}^{mm,p}_{\hat{g},h}$ from fatgraphs with h index loops on a Riemann surface of genus $\hat{g}$, where the superscript p denotes the perturbative part. The statement that the holomorphic matrix model is the string field theory of the open B-type topological string on X_{res} means that the matrix model free energy coincides with the free energy of the open B-type topological string. To be more precise, we consider the open B-type topological string on X_{res} with $\hat{N}_i$ topological branes wrapped around the i-th $\mathbb{CP}^1$ with coupling constant g_s. When mapping a Riemann surface with boundaries into the target space we know that the boundaries have to be mapped onto the holomorphic cycles. We denote the free energy for the case in which h_i boundaries are mapped to the i-th cycle by $\mathcal{F}^{oB,p}_{\hat{g},h_1,\dots h_n}$. In the corresponding matrix model with coupling constant g_s, on the other hand, one also has to choose a vacuum around which one expands to calculate the free energy. But from our analysis it is obvious that the corresponding vacuum of the matrix model is the one in which the filling fraction ν_i^* is given by the number of topological branes as

$$t\nu_i^* = \hat{N}_i g_s = \bar{S}_i \ . \tag{7.15}$$

On can now expand the matrix model around this particular vacuum (see e.g. [93], [104] for explicit examples) and from this expansion one can read off the quantities $\mathcal{F}^{mm,p}_{\hat{g},h_1,\dots h_n}$. The statement that the matrix model is the string field theory of the open B-topological string implies then that

$$\mathcal{F}^{mm,p}_{\hat{g},h_1,\dots h_n} = \mathcal{F}^{oB,p}_{\hat{g},h_1,\dots h_n} \ . \tag{7.16}$$

Let us define the quantities

$$\begin{aligned} \mathcal{F}^{mm,p}_{\hat{g}}(\bar{S}_i) &:= \sum_{h_1=1}^{\infty} \dots \sum_{h_n=1}^{\infty} \mathcal{F}^{mm,p}_{\hat{g},h_1,\dots h_n} \bar{S}_1^{h_1} \dots \bar{S}_n^{h_n} \\ &= \sum_{h_1=1}^{\infty} \dots \sum_{h_n=1}^{\infty} \mathcal{F}^{oB,p}_{\hat{g},h_1,\dots h_n} \bar{S}_1^{h_1} \dots \bar{S}_n^{h_n} =: \mathcal{F}^{oB,p}_{\hat{g}}(\bar{S}_i) \ . \end{aligned} \tag{7.17}$$

Next we look back at Eqs. (5.101) and (5.102). These are the standard special geometry relations on X_{def}, with $\mathcal{F}_0^{mm}$ as prepotential. On the other hand, the prepotential is known [81] to be the free energy of the *closed* B-type topological string at genus zero: $\mathcal{F}_0^{mm}(\bar{S}_i) = \mathcal{F}_0^B(\bar{S}_i)$. This led Dijkgraaf and Vafa to the conjecture [43] that this equality remains true for all $\hat{g}$, so that

$$\mathcal{F}^{mm}(\bar{S}_i) = \mathcal{F}^B(\bar{S}_i) \ , \tag{7.18}$$

where the left-hand side is the matrix model free energy with coupling constant g_s, and the right-hand side denotes the free energy of the *closed* B-type topological string

on X_{def} with coupling constant g_s. On the left-hand side $\bar{S}_i = g_s \hat{N}_i$, whereas on the right-hand side $\bar{S}_i = \frac{1}{4\pi^2} \int_{\Gamma_{A^i}} \Omega$.

Note that this implies that we have an open-closed duality

$$\mathcal{F}_{\hat{g}}^{B}(\bar{S}_i) = \mathcal{F}_{\hat{g}}^{mm}(\bar{S}_i) = \mathcal{F}_{\hat{g}}^{oB}(\bar{S}_i) \ . \tag{7.19}$$

In other words, we have found a closed string theory which calculates the free energy of the gauge theory, and thus we have found an example in which the old idea of 't Hooft [80] has become true.

The effective superpotential revisited

Let us finally conclude this chapter with some remarks on the effective superpotential.

So far our general philosophy has been as follows: we geometrically engineered a certain gauge theory from an open string theory on some manifold, took the manifold through a geometric transition, studied closed string theory with flux on the new manifold and found that the four-dimensional effective action generated from this string theory is nothing but the low energy effective action of the geometrically engineered gauge theory.

One might now ask whether it is also possible to find the low energy effective superpotential W_{eff} *before* going through the geometric transition from the open string setup. It was shown in [20], [130] that this is indeed possible, and that the low energy effective superpotential is schematically given by

$$W_{eff} \sim \sum_{h=1}^{\infty} \mathcal{F}_{0,h}^{oB} N h S^{h-1} + \alpha S \ , \tag{7.20}$$

where we introduced only one chiral superfield S and N D-branes (i.e. the gauge group is unbroken). To derive this formula one uses arguments that are similar to those leading to the $\mathcal{F}_{\hat{g}}(\mathcal{W}^2)^{\hat{g}}$-term in the case of the closed topological string [20]. Introducing the formal sum

$$\mathcal{F}_{\hat{g}}^{oB}(S) := \sum_{h=1}^{\infty} \mathcal{F}_{\hat{g},h} S^h \tag{7.21}$$

this can be written as

$$W_{eff} \sim N \frac{\partial \mathcal{F}_0^{oB}(S)}{\partial S} + \alpha S \ . \tag{7.22}$$

If we now use the open-closed duality (7.19) this has precisely the form of (6.33). So, in principle, one can calculate the effective superpotential from the open topological string. However, there one has to calculate all the terms $\mathcal{F}_{0,h}^{oB}$ and sum them over h. In practice this task is not feasible explicitly. The geometric transition is so useful, because it does this summation for us by mapping the sum to the quantity $\mathcal{F}_{\hat{g}}^{B}$ in closed string theory, which is much more accessible.

Chapter 8

Conclusions

Although we have covered only a small part of a vast net of interdependent theories, the picture we have drawn is amazingly rich and beautiful. We saw that string theory can be used to calculate the low energy effective superpotential, and hence the vacuum structure, of four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theories. This effective superpotential can be obtained from geometric integrals on a suitably chosen Calabi-Yau manifold, which reduce to integrals on a hyperelliptic Riemann surface. This Riemann surface also appears in the planar limit of a holomorphic matrix model, and the integrals can therefore be related to the matrix model free energy. The free energy consists of a perturbative and a non-perturbative part, and the perturbative contributions can be easily evaluated using matrix model Feynman diagrams. Therefore, after adding the non-perturbative $S \log S$ term, the effective superpotential can be obtained using matrix model perturbation theory. This is quite surprising, since vacuum expectation values like $\langle S \rangle^N = \Lambda^3$ in super Yang-Mills theory are non-perturbative in the gauge coupling. In other words, non-perturbative gauge theory quantities can be calculated from a perturbative expansion in a matrix model.

Our analysis has been rather "down to earth", in the sense that we had an explicit manifold, X_{def}, on which we had to calculate very specific integrals. After having obtained a detailed understanding of the matrix model it was not too difficult to relate our integrals to the matrix model free energy. Plugging the resulting expressions into the Gukov-Vafa-Witten formula expresses the superpotential in terms of matrix model quantities. However, this technical approach does not lead to a physical understanding of why all these theories are related, and why the Gukov-Vafa-Witten formula gives the correct superpotential. As we tried to explain in chapter 7, the deeper reason for these relations can be understood from properties of the topological string. It is well known that both the open and the closed topological string calculate terms in the four-dimensional effective action of Calabi-Yau compactifications. In particular, the low energy effective superpotential W_{eff} can be calculated by summing infinitely many quantities $\mathcal{F}^{oB}_{0,h}$ of the open topological string on X_{res}. Quite interestingly, there exists a dual closed topological string theory on X_{def}, in which this sum is captured by $\mathcal{F}^{B}_{0}$, which can be calculated from geometric integrals. The relation between the target spaces of the open and the dual closed topological string is amazingly simple

and given by a geometric transition. In a sense, this duality explains why the Gukov-Vafa-Witten formula is valid. The appearance of the holomorphic matrix model can also be understood from an analysis of the topological string, since it is nothing but the string field theory of the open topological string on X_{res}. Interestingly, the matrix model also encodes the deformed geometry, which appears once we take the large $\hat{N}$ limit.

These relations have been worked out in the articles [130], [27], [43], [44] and [45]. However, in these papers some fine points have not been discussed. In particular, the interpretation and cut-off dependence of the righthand side of

$$\int_{\Gamma_{B_i}} \Omega \sim \frac{\partial \mathcal{F}_0}{\partial S_i} \tag{8.1}$$

was not entirely clear. Over the last chapters and in [P5] we improved this situation by choosing a symplectic basis $\{\Gamma_{\alpha^i}, \Gamma_{\beta_i}, \Gamma_{\hat{\alpha}}, \Gamma_{\hat{\beta}}\}$ of the set of compact and non-compact three-cycles in X_{def}, given by $W'(x)^2 + f_0(x) + v^2 + w^2 + z^2 = 0$. These map to a basis $\{\alpha^i, \beta_i, \hat{\alpha}, \hat{\beta}\}$ of the set of relative one-cycles $H_1(\Sigma, \{Q, Q'\})$ on the Riemann surface Σ, given by $y^2 = W'(x)^2 + f_0(x)$, with two marked points Q, Q'. Then we showed that the precise form of the special geometry relations on X_{def} reads

$$-\frac{1}{2\pi i}\int_{\Gamma_{\alpha^i}} \Omega = 2\pi i \tilde{S}_i \ , \tag{8.2}$$

$$-\frac{1}{2\pi i}\int_{\Gamma_{\beta_i}} \Omega = \frac{\partial \mathcal{F}_0(t, \tilde{S})}{\partial \tilde{S}_i} \ , \tag{8.3}$$

$$-\frac{1}{2\pi i}\int_{\Gamma_{\hat{\alpha}}} \Omega = 2\pi i t \ , \tag{8.4}$$

$$-\frac{1}{2\pi i}\int_{\Gamma_{\hat{\beta}}} \Omega = \frac{\partial \mathcal{F}_0(t, \tilde{S})}{\partial t} + W(\Lambda_0) - t \log \Lambda_0^2 + o\left(\frac{1}{\Lambda_0}\right) \ . \tag{8.5}$$

where $\mathcal{F}_0(t, \tilde{S}_i)$ is the Legendre transform of the free energy of the matrix model with potential W, coupled to sources. In the last relation the integral is understood to be over the regulated cycle $\Gamma_{\hat{\beta}}$ which is an S^2-fibration over a line segment running from the n-th cut to the cut-off Λ_0. Clearly, once the cut-off is removed, the last integral diverges. These relations show that the choice of basis $\{\Gamma_{\alpha^i}, \Gamma_{\beta_i}, \Gamma_{\hat{\alpha}}, \Gamma_{\hat{\beta}}\}$, although equivalent to any other choice, is particularly useful. The integrals over the compact cycles lead to the familiar rigid special geometry relations, whereas the new features, related to the non-compactness of the manifold, only show up in the remaining two integrals. We further improved these formulae by noting that one can get rid of the polynomial divergence by introducing [P5] a paring on X_{def} defined as

$$\left\langle \Gamma_{\hat{\beta}}, \Omega \right\rangle := \int_{\Gamma_{\hat{\beta}}} (\Omega - \mathrm{d}\Phi) = (-i\pi) \int_{\hat{\beta}} (\zeta - \mathrm{d}\varphi) \ , \tag{8.6}$$

where

$$\Phi := \frac{W(x)W'(x) - \frac{2t}{n+1}x^n}{W'(x)^2 + f_0(x)} \cdot \frac{\mathrm{d}v \wedge \mathrm{d}w}{2z} \tag{8.7}$$

is such that $\int_{\Gamma_{\hat{\beta}}} \mathrm{d}\Phi = -i\pi \int_{\hat{\beta}} \mathrm{d}\varphi$, with φ as in (6.17). This pairing is very similar in structure to the one appearing in the context of relative (co-)homology and we proposed that one should use this pairing so that Eq. (8.5) is replaced by

$$-\frac{1}{2\pi i}\left\langle \Gamma_{\hat{\beta}}, \Omega \right\rangle = \frac{\partial \mathcal{F}_0(t,\tilde{S})}{\partial t} - t\log \Lambda_0^2 + o\left(\frac{1}{\Lambda_0}\right) . \tag{8.8}$$

At any rate, whether one uses this pairing or not, the integral over the non-compact cycle $\Gamma_{\hat{\beta}}$ is *not* just given by the derivative of the prepotential with respect to t, as is often claimed in the literature.

Using this pairing the modified Gukov-Vafa-Witten formula for the effective superpotential is proposed to read

$$\begin{aligned} W_{eff} &= \frac{1}{2\pi i}\sum_{i=1}^{n-1}\left(\int_{\Gamma_{\alpha^i}} G_3 \int_{\Gamma_{\beta_i}} \Omega - \int_{\Gamma_{\beta_i}} G_3 \int_{\Gamma_{\alpha^i}} \Omega\right) \\ &\quad + \frac{1}{2\pi i}\left(\int_{\Gamma_{\hat{\alpha}}} G_3 \left\langle \Gamma_{\hat{\beta}}, \Omega \right\rangle - \left\langle \Gamma_{\hat{\beta}}, G_3 \right\rangle \int_{\Gamma_{\hat{\alpha}}} \Omega\right) . \end{aligned} \tag{8.9}$$

We emphasize that, although the commonly used formula $W_{eff} \sim \int G_3 \wedge \Omega$ is very elegant, it should rather be considered as a mnemonic for (8.9) because the Riemann bilinear relations do not necessarily hold on non-compact Calabi-Yau manifolds. Note that although the introduction of the pairing did not render the integrals of Ω and G_3 over the $\Gamma_{\hat{\beta}}$-cycle finite, since they are still logarithmically divergent, these divergences cancel in (8.9) and the effective superpotential is well-defined.

The detailed analysis of the holomorphic matrix model also led to some new results. In particular, we saw that, in order to calculate the matrix model free energy from a saddle point expansion, the contour γ has to be chosen in such a way that it passes through (or at least close to) all critical points of the matrix model potential W, and the tangent vectors of γ at the critical points are such that the critical points are local minima along γ. This specific form of γ is dictated by the requirement that the planar limit spectral density $\rho_0(s)$ has to be real. ρ_0 is given by the discontinuity of $y_0 = W'(x) - 2t\omega_0$, which is one of the branches of the Riemann surface $y^2 = W'(x)^2 + f_0(x)$. The reality of ρ_0 therefore puts constraints on the coefficients in f_0 and hence on the form of the cuts in the Riemann surface. Since the curve γ has to go through all the cuts of the surface, the reality of ρ_0 constrains the form of the contour. This guarantees that one expands around a configuration for which the first derivatives of the effective action indeed vanish. To ensure that saddle points are really stable we were led to choose γ to consist of n pieces where each piece contains one cut and runs from infinity in one convergence domain to infinity of another domain. Then the "one-loop" term is a convergent, subleading Gaussian integral.

Part II

M-theory Compactifications, G_2-Manifolds and Anomalies

Chapter 9

Introduction

In the middle of the nineteen nineties it became clear that the five consistent ten-dimensional string theories, Type IIA, Type IIB, Type I, $SO(32)$-heterotic and $E_8 \times E_8$-heterotic, are not independent, but are related by duality transformations. Furthermore, a relation of these string theories to eleven-dimensional supergravity was found, and this web of interrelated theories was dubbed *M-theory* [129], [149]. One of the intriguing new features of M-theory is the appearance of an additional, eleventh dimension, which implies that the old constructions of string compactifications [31] had to be generalised. In fact, one is immediately led to the question on which manifolds one has to compactify eleven-dimensional supergravity, in order to obtain a physically interesting four-dimensional $\mathcal{N} = 1$ effective field theory. It turns out that the mechanism of these compactifications is quite similar to the one of Calabi-Yau compactifications, and the compact seven-dimensional manifold has to be a so-called *G_2-manifold*. The Kaluza-Klein reduction of eleven-dimensional supergravity on these manifolds was first derived in [115]. However, one finds that four-dimensional standard model like theories, containing non-Abelian gauge groups and charged chiral fermions, can only be obtained from G_2-compactifications if we allow the seven-manifold to be singular (see for example [6] for a review). To be more precise, if the G_2-manifold carries conical singularities, four-dimensional charged chiral fermions occur which are localised at these singularities. Non-Abelian gauge groups arise from ADE-singularities on the G_2-manifold. Clearly, once a theory with charged chiral fermions is constructed, one has to check whether it is also free of anomalies. Two different notions of anomaly cancellation occur in this context. *Global anomaly cancellation* basically is the requirement that the four-dimensional theory is anomaly free after summation of the anomaly contributions from all the singularities of the internal manifold. *Local anomaly cancellation* on the other hand imposes the stronger condition that the contributions to the anomalies associated with each singularity have to cancel separately. We will study these issues in more detail below. What we find is that, in the case of singular G_2-compactifications, the anomalies present at a given singularity are cancelled locally by a contribution which "flows" into the singularity from the bulk, provided one modifies the fields close to the singularity [152], [P2].

Compactifications on G_2-manifolds lead to four-dimensional Minkowski space, since

the metric on a G_2-manifold is Ricci-flat. There are also other solutions of the equations of motion of eleven-dimensional supergravity. One of them is the direct product of AdS_4 with a seven-dimensional compact Einstein space with positive curvature. Since the metric on the seven-manifold is Einstein rather than Ricci-flat, these manifolds cannot be G_2-manifolds. However, a generalisation of the concept of G_2-manifolds to the case of Einstein manifolds exists. These manifolds are known to be *weak G_2-manifolds*. Quite interestingly, we were able to write down for the first time a family of explicit metrics for such weak G_2-manifolds that are compact and have two conical singularities [P1]. Although these manifolds have weak G_2 rather than G_2 metrics, they are quite similar to G_2-manifolds, and hence provide a framework in which many of the features of compact, singular G_2-manifolds can be studied explicitly.

Another context in which the cancellation of anomalies plays a crucial role is the M5-brane. It carries chiral fields on its six-dimensional world-volume and this field theory on its own would be anomalous. However, once embedded into eleven-dimensional supergravity one finds that a contribution to the anomaly flows from the bulk into the brane, exactly cancelling the anomaly. In fact, from these considerations a first correction term to eleven-dimensional supergravity has been deduced already ten years ago [48], [150], [60]. The mechanism of anomaly cancellation for the M5-brane has been reviewed in detail in [P3] and [P4] and we will not cover it here.

Finally, our methods of local anomaly cancellation and inflow from the bulk can be applied to eleven-dimensional supergravity on the interval [83], [P3]. In this context an intriguing interplay of new degrees of freedom living on the boundaries, a modified Bianchi identity and anomaly inflow leads to a complete cancellation of anomalies. The precise mechanism has been the subject of quite some controversy in the literature (see for example [22] for references). Our treatment in [P3] finally provides a clear proof of *local* anomaly cancellation.

In the following I am going to make this discussion more precise by summarising the results of the publications [P1], [P2] and [P3]. The discussion will be rather brief, since many of the details can be found in my work [P4]. In the remainder of the introduction I quickly review the concept of G_2-manifolds, explain the action of eleven-dimensional supergravity and introduce the important concept of anomaly inflow. The notation is explained in appendix A.

G_2-manifolds

Definition 9.1 Let $(x_1, \dots, x_7)$ be coordinates on $\mathbb{R}^7$. Write $\mathrm{d}\mathbf{x}_{ij\dots l}$ for the exterior form $\mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \dots \wedge \mathrm{d}x_l$ on $\mathbb{R}^7$. Define a three-form Φ_0 on $\mathbb{R}^7$ by

$$\Phi_0 := \mathrm{d}\mathbf{x}_{123} + \mathrm{d}\mathbf{x}_{516} + \mathrm{d}\mathbf{x}_{246} + \mathrm{d}\mathbf{x}_{435} + \mathrm{d}\mathbf{x}_{147} + \mathrm{d}\mathbf{x}_{367} + \mathrm{d}\mathbf{x}_{257} \ . \tag{9.1}$$

The subgroup of $GL(7, \mathbb{R})$ preserving Φ_0 is the exceptional Lie group G_2. It is compact, connected, simply connected, semisimple and 14-dimensional, and it also fixes the four-form

$$*\Phi_0 = \mathrm{d}\mathbf{x}_{4567} + \mathrm{d}\mathbf{x}_{2374} + \mathrm{d}\mathbf{x}_{1357} + \mathrm{d}\mathbf{x}_{1276} + \mathrm{d}\mathbf{x}_{2356} + \mathrm{d}\mathbf{x}_{1245} + \mathrm{d}\mathbf{x}_{1346} \ , \tag{9.2}$$

the Euclidean metric $g_0 = dx_1^2 + \dots dx_7^2$, and the orientation on $\mathbb{R}^7$.

Definition 9.2 A *G_2-structure* on a seven-manifold M is a principal subbundle of the frame bundle of M with structure group G_2. Each G_2-structure gives rise to a 3-form Φ and a metric g on M, such that every tangent space of M admits an isomorphism with $\mathbb{R}^7$ identifying Φ and g with Φ_0 and g_0, respectively. We will refer to (Φ, g) as a G_2-structure. Let ∇ be the Levi-Civita connection, then $\nabla\Phi$ is called the *torsion* of (Φ, g). If $\nabla\Phi = 0$ then (Φ, g) is called *torsion free.* A *G_2-manifold* is defined as the triple (M, Φ, g), where M is a seven-manifold, and (Φ, g) a torsion-free G_2-structure on M.

Proposition 9.3 *Let M be a seven-manifold and (Φ, g) a G_2-structure on M. Then the following are equivalent:*

(i) $\nabla\Phi = 0$,
(ii) $\mathrm{d}\Phi = \mathrm{d} * \Phi = 0$,
(iii) $\mathrm{Hol}(g) \subseteq G_2$.

Note that the holonomy group of a G_2-manifold may be a proper subset of G_2. However, we will mean a manifold with holonomy group G_2 whenever we speak of a G_2-manifold in the following. Let us list some properties of compact Riemann manifolds (M, g) with $\mathrm{Hol}(g) = G_2$.

- M is a spin manifold and there exists exactly one covariantly constant spinor,[1] $\nabla^S\theta = 0$.
- g is Ricci-flat.
- The Betti numbers are $b^0 = b^7 = 1$, $b^1 = b^6 = 0$ and $b^2 = b^5$ and $b^3 = b^4$ arbitrary.

Many more details on G_2-manifolds can be found in [P4]. A thorough mathematical treatment of G_2-manifolds, which also contains the proof of proposition 9.3 can be found in [85].

Eleven-dimensional supergravity

It is current wisdom in string theory [149] that the low energy limit of M-theory is eleven-dimensional supergravity [39]. Therefore, some properties of M-theory can be deduced from studying this well understood supergravity theory. Here we review the basic field content, the Lagrangian and its equations of motion. More details can be found in [135], [47], [49] and [140]. For a recent review see [108]. The field content of eleven-dimensional supergravity is remarkably simple. It consists of the metric g_{MN}, a Majorana spin-$\frac{3}{2}$ fermion ψ_M and a three-form $C = \frac{1}{3!}C_{MNP}\mathrm{d}z^M \wedge \mathrm{d}z^N \wedge \mathrm{d}z^P$, where z^M is a set of coordinates on the space-time manifold M_{11}. These fields can be combined

[1] ∇^S contains the spin connection, see (9.6) or appendix A.

to give the unique $\mathcal{N}=1$ supergravity theory in eleven dimensions. The full action is[2]

$$\begin{aligned} S &= \frac{1}{2\kappa_{11}^2}\int [\mathcal{R} * 1 - \frac{1}{2}G\wedge *G - \frac{1}{6}C\wedge G\wedge G] \\ &\quad + \frac{1}{2\kappa_{11}^2}\int d^{11}z\sqrt{g}\bar{\psi}_M\Gamma^{MNP}\nabla_N^S\left(\frac{\omega+\hat{\omega}}{2}\right)\psi_P \\ &\quad - \frac{1}{2\kappa_{11}^2}\frac{1}{192}\int d^{11}z\sqrt{g}\left(\bar{\psi}_M\Gamma^{MNPQRS}\psi_N + 12\bar{\psi}^P\Gamma^{RS}\psi^Q\right)\left(G_{PQRS}+\hat{G}_{PQRS}\right) . \end{aligned} \tag{9.3}$$

To explain the contents of the action, we start with the commutator of the vielbeins, which defines the *anholonomy coefficients* $\Omega_{AB}{}^C$

$$[e_A, e_B] = [e_A{}^M\partial_M, e_B{}^N\partial_N] = \Omega_{AB}{}^C e_C . \tag{9.4}$$

Relevant formulae for the spin connection are

$$\begin{aligned} \omega_{MAB}(e) &= \frac{1}{2}(-\Omega_{MAB}+\Omega_{ABM}-\Omega_{BMA}) , \\ \omega_{MAB} &= \omega_{MAB}(e) + \frac{1}{8}[-\bar{\psi}_P\Gamma_{MAB}{}^{PQ}\psi_Q + 2(\bar{\psi}_M\Gamma_B\psi_A - \bar{\psi}_M\Gamma_A\psi_B + \bar{\psi}_B\Gamma_M\psi_A)] , \\ \hat{\omega}_{MAB} &:= \omega_{MAB} + \frac{1}{8}\bar{\psi}_P\Gamma_{MAB}{}^{PQ}\psi_Q . \end{aligned} \tag{9.5}$$

ψ_M is a Majorana vector-spinor. The Lorentz covariant derivative reads

$$\nabla_M^S(\omega)\psi_N := \partial_M\psi_N + \frac{1}{4}\omega_{MAB}\Gamma^{AB}\psi_N . \tag{9.6}$$

For further convenience we set

$$\widetilde{\nabla}_M^S(\omega)\psi_N := \nabla_M^S(\omega)\psi_N - \frac{1}{288}\left(\Gamma_M{}^{PQRS} - 8\delta_M^P\Gamma^{QRS}\right)\hat{G}_{PQRS}\psi_N . \tag{9.7}$$

$$G := dC \quad \text{i.e.} \quad G_{MNPQ} = 4\partial_{[M}C_{NPQ]} . \tag{9.8}$$

$\hat{G}_{MNPQ}$ is defined as

$$\hat{G}_{MNPQ} := G_{MNPQ} + 3\bar{\psi}_{[M}\Gamma_{NP}\psi_{Q]} . \tag{9.9}$$

The action is invariant under the supersymmetry transformations

$$\begin{aligned} \delta e^A{}_M &= -\frac{1}{2}\bar{\eta}\Gamma^A\psi_M , \\ \delta C_{MNP} &= -\frac{3}{2}\bar{\eta}\Gamma_{[MN}\psi_{P]} , \\ \delta\psi_M &= \widetilde{\nabla}_M^S(\hat{\omega})\eta . \end{aligned} \tag{9.10}$$

[2]We define $\bar{\psi}_M := i\psi_M^\dagger\Gamma^0$, see appendix A.

Next we turn to the equations of motion. We will only need solutions of the equations of motion with the property that $\psi_M \equiv 0$. Since ψ_M appears at least bilinearly in the action, we can set ψ_M to zero before varying the action. This leads to an enormous simplification of the calculations. The equations of motion with vanishing fermion field read

$$\begin{aligned}\mathcal{R}_{MN}(\omega) - \frac{1}{2} g_{MN}\mathcal{R}(\omega) &= \frac{1}{12}\left(G_{MPQR}G_N{}^{PQR} - \frac{1}{8} g_{MN} G_{PQRS}G^{PQRS}\right) ,\\ d * G + \frac{1}{2} G \wedge G &= 0 .\end{aligned} \tag{9.11}$$

In addition to those field equations we also know that G is closed, as it is exact,

$$dG = 0 . \tag{9.12}$$

A solution $(M, \langle g\rangle, \langle C\rangle, \langle\psi\rangle)$ of the equations of motion is said to be *supersymmetric* if the variations (9.10) vanish at the point $e^A{}_M = \langle e^A{}_M\rangle$, $C_{MNP} = \langle C_{MNP}\rangle$, $\psi_M = \langle\psi_M\rangle$. All the vacua we are going to study have vanishing fermionic background, $\langle\psi_M\rangle = 0$, so the first two equations are trivially satisfied and the last one reduces to

$$\widetilde{\nabla}^S_M(\omega)\eta = 0 , \tag{9.13}$$

evaluated at $C_{MNP} = \langle C_{MNP}\rangle$, $e^A{}_M = \langle e^A{}_M\rangle$ and $\psi_M = 0$. We see that $e^A{}_M$ and C_{MNP} are automatically invariant and we find that the vacuum is supersymmetric if and only if there exists a spinor η s.t. $\forall M$

$$\nabla^S_M\eta - \frac{1}{288}\left(\Gamma_M{}^{PQRS} - 8\delta^P_M\Gamma^{QRS}\right) G_{PQRS}\eta = 0 . \tag{9.14}$$

Solutions of the equations of motion

Given the explicit form of the equations of motion, it is easy to see that $\langle\psi_M\rangle = 0$, $\langle C\rangle = 0$, together with any Ricci-flat metric on the base manifold M_{11} is a solution. In particular, this is true for $(M_{11}, g) = (\mathbb{R}^4 \times M, \eta \times g)$, where $(\mathbb{R}^4, \eta)$ is Minkowski space and (M, g) is a G_2-manifold. For such a vacuum the condition (9.14), reduces to

$$\nabla^S\eta = 0 . \tag{9.15}$$

The statement that the effective four-dimensional theory should be $\mathcal{N} = 1$ supersymmetric translates to the requirement that (9.15) has exactly four linearly independent solutions. After the compactification the original Poincaré group $P(10, 1)$ is broken to $P(3, 1) \times P(7)$. The **32** of $SO(10, 1)$ decomposes as $\mathbf{32} = \mathbf{4} \otimes \mathbf{8}$, thus, for a spinor in the compactified theory we have

$$\eta(x, y) = \epsilon(x) \otimes \theta(y) , \tag{9.16}$$

with ϵ a spinor in four and θ a spinor in seven dimensions. The Γ-matrices can be rewritten as

$$\begin{aligned} \Gamma^a &= \gamma^a \otimes \mathbb{1} \ , \\ \Gamma^m &= \gamma_5 \otimes \gamma^m \ , \end{aligned} \tag{9.17}$$

with $\{\gamma^m\}$ the generators of a Clifford algebra in seven dimensions. Then it is not hard to see that for $\nabla^S = \nabla^S_M dz^M$ one has

$$\nabla^S = \nabla^S_4 \otimes \mathbb{1} + \mathbb{1} \otimes \nabla^S_7 \ . \tag{9.18}$$

Therefore, (9.15) reads

$$(\nabla^S_4 \otimes \mathbb{1} + \mathbb{1} \otimes \nabla^S_7)\epsilon(x) \otimes \theta(y) = \nabla^S_4 \epsilon(x) \otimes \theta(y) + \epsilon(x) \otimes \nabla^S_7 \theta(y) = 0 \ . \tag{9.19}$$

On Minkowski space we can find a basis of four constant spinors ϵ^i. The condition we are left with is

$$\nabla^S_7 \theta(y) = 0 \ . \tag{9.20}$$

Thus, the number of solutions of (9.15) is four times the number of covariantly constant spinors on the compact seven manifold. Since we already saw that a G_2-manifold carries precisely one covariantly constant spinor, we just proved our statement that compactifications on G_2-manifolds lead to four-dimensional $\mathcal{N} = 1$ theories.

Next consider what is known as the *Freund-Rubin solution* of eleven-dimensional gravity. Here (M_{11}, g) is given by a Riemannian product, $(M_{11}, g) = (M_4 \times M_7, g_1 \times g_2)$, and [3]

$$\begin{aligned} M_{11} &= S^1 \times \mathbb{R}^3 \times M_7, \quad M_7 \text{ compact} \ , \\ \langle \psi_M \rangle &= 0 \ , \\ \langle g_1 \rangle &= g(AdS_4) \ , \\ \langle g_2 \rangle & \quad \text{Einstein, s.t. } \mathcal{R}_{mn} = \tfrac{1}{6} f^2 \langle g_{2mn} \rangle \ , \\ \langle G_{\mu\nu\rho\sigma} \rangle &= f \sqrt{\langle g_1 \rangle}\, \widetilde{\epsilon}_{\mu\nu\rho\sigma} \ . \end{aligned} \tag{9.21}$$

Next we want to analyze the consequences of (9.14) for the Freund-Rubin solutions. We find

$$\begin{aligned} \nabla^S_\mu \eta &= -\frac{if}{6}(\gamma_\mu \gamma_5 \otimes \mathbb{1})\eta \ , \\ \nabla^S_m \eta &= \frac{if}{12}(\mathbb{1} \otimes \gamma_m)\eta \ . \end{aligned} \tag{9.22}$$

Again we have the decomposition $\mathbf{32} = \mathbf{4} \otimes \mathbf{8}$ and hence $\eta(x, y) = \epsilon(x) \otimes \theta(y)$. Then

[3] Recall that the topology of AdS_4 is $S^1 \times \mathbb{R}^3$.

(9.22) reduce to

$$\nabla^S_\mu \epsilon = -\frac{if}{6}\gamma_\mu\gamma_5\epsilon, \tag{9.23}$$

$$\nabla^S_m \theta = \frac{if}{12}\gamma_m\theta. \tag{9.24}$$

On AdS_4 one can find four spinors satisfying (9.23). Therefore, the number of spinors η, satisfying (9.22) is four times the number of spinors θ which are solutions of (9.24). In other words, to find Freund-Rubin type solutions with $\mathcal{N} = k$ supersymmetry we need to find compact seven-dimensional Einstein spaces with positive curvature and exactly k Killing spinors. One possible space is the seven-sphere which admits eight Killing spinors, leading to maximal supersymmetry in four dimensions. A seven-dimensional Einstein manifold with exactly one Killing spinor is known as a *weak G_2-manifold*.

Kaluza-Klein compactification on a smooth G_2-manifolds

We already mentioned that one has to introduce singularities into the compact G_2-manifold in order to generate interesting physics. Indeed, for smooth G_2-manifolds we have the following proposition.

Proposition 9.4 *The low energy effective theory of M-theory on $(\mathbb{R}^4 \times X, \eta \times g)$ with (X, g) a smooth G_2-manifold is an $\mathcal{N} = 1$ supergravity theory coupled to $b^2(X)$ Abelian vector multiplets and $b^3(X)$ massless neutral chiral multiplets.*

This field content was determined in [115], the Kaluza-Klein compactification procedure is reviewed in [P4]. Note that although there are chiral fields in the effective theory these are not very interesting, since they do not couple to the gauge fields.

Anomaly inflow

Before we embark on explaining the details of the mechanism of anomaly cancellation on singular G_2-manifolds, we want to comment on a phenomenon known as anomaly inflow. A comprehensive discussion of anomalies and many references can be found in [P4], the most important results are listed in appendix E, to which we refer the reader for further details. The concept of anomaly inflow in effective theories was pioneered in [30] and further studied in [110]. See [76] for a recent review. Here we analyse the extension of these ideas to the context of M-theory, as studied in [P3].

Consider a theory in $d = 2n$ dimensions containing a massless fermion ψ coupled to a non-Abelian external gauge field $A = A_a T_a$ with gauge invariant (Euclidean) action $S^E[\psi, A]$. The current

$$J^M_a(x) := \frac{\delta S^E[\psi, A]}{\delta A_{aM}(x)}\ , \tag{9.25}$$

is conserved, because of the gauge invariance of the action, $D_M J^M_a(x) = 0$. Next we define the functional

$$\exp\left(-X[A]\right) := \int D\psi D\bar{\psi} \exp\left(-S^E[\psi, A]\right) \ . \tag{9.26}$$

Under a gauge variation $A(x) \to A'(x) = A(x) + D\epsilon(x)$ with $\epsilon(x) = \epsilon_a(x) T_a$ the Euclidean action is invariant, but, in general, the measure transforms as

$$D\psi D\bar{\psi} \to \exp\left(i \int (\mathrm{d}^d x)_E \epsilon_a(x) G_a[x; A]\right) D\psi D\bar{\psi} \ . \tag{9.27}$$

Here $(d^d x)_E$ is the Euclidean measure, and the quantity $G_a[x;A]$, called the anomaly function, depends on the theory under consideration (see [P4] for some explicit examples). Variation of (9.26) then gives

$$\exp(-X[A]) \int (d^d x)_E D_M \langle J^M_a(x) \rangle \epsilon_a(x) = \int (d^d x)_E \int D\psi D\bar{\psi} [i G_a[x; A] \epsilon_a(x)] \exp(-S^E) \ ,$$

where we used the invariance of the Euclidean action S^E under local gauge transformations. Therefore,

$$D_M \langle J^M_a(x) \rangle = i G_a[x; A] \ , \tag{9.28}$$

and we find that the quantum current $\langle J^M_a(x) \rangle$ is not conserved. The anomaly $G_a[x;A]$ can be evaluated from studying the transformation properties of the path integral measure.

Note that (9.27) implies

$$\delta X = -i \int (\mathrm{d}^d x)_E \epsilon_a(x) G_a[x; A] =: i \int I^1_{2n} \ , \tag{9.29}$$

where we defined a $2n$-form I^1_{2n}. We see that a theory is free of anomalies if the variation of the functional X vanishes. This variation is captured by the form I^1_{2n} and it would be nice if we could find a simple way to derive this form for a given theory. This is in fact possible, as explained in some detail in the appendix. It turns out [126], [153], [102] that the $2n$-form I^1_{2n} is related to a $2n+2$-form I_{2n+2} via the so called decent equations,

$$\mathrm{d}I^1_{2n} = \delta I_{2n+1} \quad , \quad \mathrm{d}I_{2n+1} = I_{2n+2} \ , \tag{9.30}$$

where I_{2n+2} is a polynomial in the field strengths. Furthermore, the anomaly polynomial I_{2n+2} depends only on the field content of the theory. It can be shown that the only fields leading to anomalies are spin-$\frac{1}{2}$ fermions, spin-$\frac{3}{2}$ fermions and forms with (anti-)self-dual field strength. The anomaly polynomials corresponding to these fields are given by (see [12], [11] and references therein)

$$I^{(1/2)}_{2n+2} = -2\pi \left[\hat{A}(M_{2n}) \ \mathrm{ch}(F)\right]_{2n+2} \ , \tag{9.31}$$

$$I^{(3/2)}_{2n+2} = -2\pi \left[\hat{A}(M_{2n}) \ \left(\mathrm{tr} \ \exp\left(\frac{i}{2\pi} R\right) - 1\right) \ \mathrm{ch}(F)\right]_{2n+2} \ , \tag{9.32}$$

$$I^{A}_{2n+2} = -2\pi \left[\left(-\frac{1}{2}\right) \frac{1}{4} \ L(M_{2n})\right]_{2n+2} \ . \tag{9.33}$$

To be precise these are the anomalies of spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ particles of positive chirality and a self-dual form in Euclidean space, under the gauge transformation $\delta A = D\epsilon$ and the local Lorentz transformations $\delta\omega = D\epsilon$. All the quantities appearing in these formulae are explained in appendices B.4 and E. The polynomials of the spin-$\frac{1}{2}$ fields, $I^{(1/2)}_{2n+2}$, can be written as a sum of terms containing only the gauge fields, terms containing only the curvature tensor and terms containing both. These terms are often referred to as the gauge, the gravitational and the mixed anomaly, respectively. As to determine whether a theory is anomalous or not, is is then sufficient to add all the anomaly polynomials I_{2n+2}. If they sum up to zero, the variation of the quantum effective action (9.29) vanishes[4] as well, and the theory is anomaly free.

It turns out that this formalism has to be generalized, since we often encounter problems in M-theory in which the classical action is not fully gauge invariant. One might argue that in this case the term "anomaly" loses its meaning, but this is in fact not true. The reason is that in many cases we study theories on manifolds with boundary which are gauge invariant in the bulk, but the non-vanishing boundary contributes to the variation of the action. So in a sense, the variation is nonzero because of global geometric properties of a given theory. If we studied the same Lagrangian density on a more trivial manifold, the action would be perfectly gauge invariant. This is why it still makes sense to speak of an anomaly. Of course, if we vary the functional (9.26) in theories which are not gauge invariant we obtain an additional contribution on the right-hand side. This contribution is called an *anomaly inflow term*, for reasons which will become clear presently.

Consider for example a theory which contains the topological term of eleven-dimensional supergravity. In fact, all the examples we are going to study involve either this term or terms which can be treated similarly. Clearly, $\int_{M_{11}} C \wedge dC \wedge dC$ is invariant under $C \to C + \mathrm{d}\Lambda$ as long as M_{11} has no boundary. In the presence of a boundary we get the non-vanishing result $\int_{\partial M_{11}} \Lambda \wedge dC \wedge dC$. Let us study what happens in such a case to the variation of our functional. To do so we first need to find out how our action can be translated to Euclidean space. The rules are as follows (see [P3] for a detailed discussion of the transition from Minkowski to Euclidean space)

$$\begin{aligned}
x^1_E &:= ix^0_M \ , \quad x^2_E := x^1_M \ , \ldots \\
(d^{11}x)_E &:= id^{11}x \ , \\
C^E_{1MN} &:= -iC_{0(M-1)(N-1)} \quad , \quad M, N \ldots \in \{2, \ldots, 11\} \\
C^E_{MNP} &= C_{(M-1)(N-1)(P-1)} \ , \\
\tilde{\epsilon}^E_{123\ldots 11} &= +1 \ .
\end{aligned} \tag{9.34}$$

[4]This is in fact not entirely true. It can happen that the sum of the polynomials I_{2n+2} of a given theory vanishes, but the variation of X is non-zero. However, in these cases one can always add a local counterterm to the action, such that the variation of X corresponding to the modified action vanishes.

We know that $S^M = iS^E$, where S^M is the Minkowski action, but explicitly we have[5]

$$\begin{aligned}
S^M_{kin} &= -\frac{1}{4\kappa_{11}^2}\int d^{11}x\sqrt{g}\,\frac{1}{4!}G_{MNPQ}G^{MNPQ} \\
&= \frac{i}{4\kappa_{11}^2}\int (d^{11}x)_E\sqrt{g}\,\frac{1}{4!}G^E_{MNPQ}(G^E)^{MNPQ}\ , \\
S^M_{top} &= \frac{1}{12\kappa_{11}^2}\int d^{11}x\sqrt{g}\,\frac{1}{3!4!4!}\epsilon^{M_0\dots M_{10}}C_{M_0M_1M_2}G_{M_3M_4M_5M_6}G_{M_7M_8M_9M_{10}} \\
&= -\frac{1}{12\kappa_{11}^2}\int (d^{11}x)_E\sqrt{g}\,\frac{1}{3!4!4!}(\epsilon^E)^{M_1\dots M_{11}}C^E_{M_1M_2M_3}G^E_{M_4M_5M_6M_7}G^E_{M_8M_9M_{10}M_{11}}\ .
\end{aligned} \tag{9.35}$$

But then we can read off

$$\begin{aligned}
S^E &= \frac{1}{4\kappa_{11}^2}\int (d^{11}x)_E\sqrt{g}\,\frac{1}{4!}G^E_{MNPQ}(G^E)^{MNPQ} \\
&\quad +\frac{i}{12\kappa_{11}^2}\int (d^{11}x)_E\sqrt{g}\,\frac{1}{3!4!4!}(\epsilon^E)^{M_1\dots M_{11}}C^E_{M_1M_2M_3}G^E_{M_4M_5M_6M_7}G^E_{M_8M_9M_{10}M_{11}}\ ,
\end{aligned}$$

where a crucial factor of i turns up. We write $S^E = S^E_{kin} + S^E_{top} =: S^E_{kin} + i\widetilde{S}^E_{top}$, because S^E_{top} is imaginary, so $\widetilde{S}^E_{top}$ is real. Then, for an eleven-manifold with boundary, we find $\delta S^E = i\delta\tilde{S}^E_{top} = \frac{i}{12\kappa_{11}^2}\int_{\partial M_{11}}\Lambda^E\wedge G^E\wedge G^E$. But this means that δS^E has precisely the right structure to cancel an anomaly on the ten-dimensional space ∂M_{11}. This also clarifies why one speaks of anomaly inflow. A contribution to an anomaly on ∂M_{11} is obtained by varying an action defined in the bulk M_{11}.

Clearly, whenever one has a theory with $\delta S^M \neq 0$ we find the master formula

$$\delta X = \delta S^E + i\int I^1_{2n} = -i\delta S^M + i\int I^1_{2n}\ . \tag{9.36}$$

The theory is anomaly free if and only if the right-hand side vanishes. In other words, to check whether a theory is free of anomalies we have to rewrite the action in Euclidean space, calculate its variation and the corresponding $2n+2$-form and add i times the anomaly polynomials corresponding to the fields present in the action. If the result vanishes the theory is free of anomalies. In doing so one has to be careful, however, since the translation from Minkowski to Euclidean space is subtle. In particular, one has to keep track of the chirality of the particles involved. The reason is that with our conventions (A.27) for the matrix Γ_{d+1} we have $\Gamma^E_{d+1} = -\Gamma^M_{d+1}$ for $d = 4k+2$, but $\Gamma^E_{d+1} = \Gamma^M_{d+1}$ for $4k$. In other words a fermion of positive chirality in four dimensional Minkowski space translates to one with positive chirality in four-dimensional Euclidean space. In six or ten dimensions, however, the chirality changes. To calculate the anomalies one has to use the polynomials after having translated everything to Euclidean space.

[5] Our conventions are such that $\epsilon^{M_1\dots M_d} = \mathrm{sig}(g)\frac{1}{\sqrt{g}}\tilde{\epsilon}^{M_1\dots M_d}$, and $\tilde{\epsilon}^{M_1\dots M_d}$ is totally anti-symmetric with $\tilde{\epsilon}^{01\dots d} = +1$. See [P3] and appendix A for more details.

Chapter 10

Anomaly Analysis of M-theory on Singular G_2-Manifolds

It was shown in [17], [7] that compactifications on G_2-manifolds can lead to charged chiral fermions in the low energy effective action, if the compact manifold has a conical singularity. Non-Abelian gauge fields arise [3], [4] if we allow for ADE singularities on a locus Q of dimension three in the G_2-manifold. We will not review these results but refer the reader to the literature [6]. We are more interested in the question whether, once the manifold carries conical singularities, the effective theory is free of anomalies. This chapter is based on the results of [P2].

10.1 Gauge and mixed anomalies

Let then X be a compact G_2-manifold that is smooth except for conical singularities[1] P_α, with α a label running from one to the number of singularities in X. Then there are chiral fermions sitting at a given singularity P_α. They have negative[2] chirality and are charged under the gauge group $U(1)^{b^2(X)}$ (c.f. proposition 9.4). Their contribution to the variation of X is given by

$$\delta X|_{anomaly} = iI_4^1 \quad \text{with} \quad I_6 = -2\pi[(-1)\hat{A}(M_4)\text{ch}(F)]_6 \ . \tag{10.1}$$

Here the subscript "anomaly" indicates that these are the contributions to δX coming from a variation of the measure. Later on, we will have to add a contribution coming from the variation of the Euclidean action. The sign of I_6 is differs from the one in (9.31) because the fermions have negative chirality. The anomaly polynomials corresponding to gauge and mixed anomalies localized at P_α are then given by (see appendices A and

[1]Up to now it is not clear whether such G_2-manifolds exist, however, examples of non-compact spaces with conical singularities are known [63], and compact weak G_2-manifolds with conical singularities were constructed in [P1].

[2]Recall that this is true both in Euclidean and in Minkowski space, since $\gamma_5^E = \gamma_5^M$, such that the chirality does not change if we translate from Minkowski to Euclidean space.

E for the details, in particular $F = iF^i q^i$),

$$I_\alpha^{(gauge)} = -\frac{1}{(2\pi)^2 3!} \sum_{\sigma \in T_\alpha} \left(\sum_{i=1}^{b^2(X)} q_\sigma^i F^i \right)^3 \quad , \quad I_\alpha^{(mixed)} = \frac{1}{24} \sum_{\sigma \in T_\alpha} \left(\sum_{i=1}^{b^2(X)} q_\sigma^i F^i \right) p_1' \, . \tag{10.2}$$

σ labels the four dimensional chiral multiplets Φ_σ which are present at the singularity P_α. T_α is simply a set containing all these labels. q_σ^i is the charge of Φ_σ with respect to the i-th gauge field A^i. As all the gauge fields come from a Kaluza-Klein expansion of the C-field we have $b^2(X)$ of them. $p_1' = -\frac{1}{8\pi^2} \operatorname{tr} R \wedge R$ is the first Pontrjagin class of four dimensional space-time $\mathbb{R}^4$. Our task is now to cancel these anomalies locally, i.e. separately at each singularity.

So far we have only been using eleven-dimensional supergravity, the low energy limit of M-theory. However, in the neighbourhood of a conical singularity the curvature of X blows up. Close to the singularity P_α the space X is a cone on some manifold Y_α (i.e. close to P_α we have $\mathrm{d}s_X^2 \simeq \mathrm{d}r_\alpha^2 + r_\alpha^2 \mathrm{d}s_{Y_\alpha}^2$). But as X is Ricci-flat Y_α has to be Einstein with $\mathcal{R}_{mn}^{Y_\alpha} = 5\delta_{mn}$. The Riemann tensor on X and Y_α are related by $R^{Xmn}{}_{pq} = \frac{1}{r_\alpha^2}(R^{Y_\alpha mn}{}_{pq} - \delta_p^m \delta_q^n + \delta_q^m \delta_p^n)$, for $m \in \{1, 2, \ldots, 6\}$. Thus, the supergravity description is no longer valid close to a singularity and one has to resort to a full M-theory calculation, a task that is currently not feasible.

To tackle this problem we use an idea that has first been introduced in [60] in the context of anomaly cancellation on the M5-brane. The world-volume W_6 of the M5-brane supports chiral field which lead to an anomaly. Quite interestingly, one can cancel these anomalies using the inflow mechanism, but only if the topological term of eleven-dimensional supergravity is modified in the neighbourhood of the brane. Since we will proceed similarly below, let us quickly motivate these modifications. The five-brane acts as a source for the field G, i.e. the Bianchi identity, $\mathrm{d}G = 0$, is modified to $\mathrm{d}G \sim \delta^{(5)}(W_6)$. In the treatment of [60] a small neighbourhood of the five-brane world-volume W_6 is cut out, creating a boundary. Then one introduces a smooth function ρ which is zero in the bulk but drops to -1 close to the brane, in such a way that $\mathrm{d}\rho$ has support only in the neighbourhood of the boundary. This function is used to smear out the Bianchi identity by writing it as $\mathrm{d}G \sim \ldots \wedge \mathrm{d}\rho$. The solution to this identity is given by $G = \mathrm{d}C + \ldots \wedge \mathrm{d}\rho$, i.e. the usual identity $G = \mathrm{d}C$ is corrected by terms localised on the boundary. Therefore, it is not clear a priori how the topological term of eleven-dimensional supergravity should be formulated (since for example the terms $C \wedge \mathrm{d}C \wedge \mathrm{d}C$ and $C \wedge G \wedge G$ are now different). It turns out that all the anomalies of the M5-brane cancel if the topological term reads $\widetilde{S}_{\mathrm{CS}} = -\frac{1}{12\kappa_{11}^2} \int \widetilde{C} \wedge \widetilde{G} \wedge \widetilde{G}$, where $\widetilde{C}$ is a field that is equal to C far from the brane, but is modified in the neighbourhood of the brane. Furthermore, $\widetilde{G} = \mathrm{d}\widetilde{C}$. This mechanism of anomaly cancellation in the context of the M5-brane is reviewed in detail in [P3] and [P4].

Now we show that a similar treatment works for conical singularities. We first concentrate on the neighbourhood of a given conical singularity P_α with a metric locally given by $\mathrm{d}s_X^2 \simeq \mathrm{d}r_\alpha^2 + r_\alpha^2 \mathrm{d}s_{Y_\alpha}^2$. The local radial coordinate obviously is $r_\alpha \geq 0$, the singularity being at $r_\alpha = 0$. As mentioned above, there are curvature invariants

of X that diverge as $r_\alpha \to 0$. In particular, supergravity cannot be valid down to $r_\alpha = 0$. Motivated by the methods used in the context of the M5-brane, we want to modify our fields close to the singularity. More precisely, we want to cut of the fluctuating fields using a smooth function ρ, which equals one far from the singularity but is zero close to it. The geometry itself is kept fixed, and in particular we keep the metric and curvature on X. Said differently, we cut off all fields that represent the quantum fluctuations, but keep the background fields (in particular the background geometry) as before. To be specific, we introduce a small but finite regulator ϵ, and the regularised step function ρ_α such that

$$\rho_\alpha(r_\alpha) = \begin{cases} 0 & \text{for} \quad 0 \le r_\alpha \le R - \epsilon \\ 1 & \text{for} \quad \quad r_\alpha \ge R + \epsilon \end{cases} \tag{10.3}$$

where ϵ/R is small. Using a partition of unity we can construct a smooth function ρ on X from these ρ_α in such a way that ρ vanishes for points with a distance to a singularity which is less than $R - \epsilon$ and is one for distances larger than $R + \epsilon$. We denote the points of radial coordinate R in the chart around P_α by Y_α, where the orientation of Y_α is defined in such a way that its normal vector points away from the singularity. All these conventions are chosen in such a way that $\int_X (\ldots) \wedge d\rho = \sum_\alpha \int_{Y_\alpha} (\ldots)$. The shape of the function ρ is irrelevant, in particular, one might use ρ^2 instead of ρ, i.e. $\rho^n \simeq \rho$. However, when evaluating integrals one has to be careful since $\rho^n \mathrm{d}\rho = \frac{1}{n+1} \mathrm{d}\rho^{n+1} \simeq \frac{1}{n+1} \mathrm{d}\rho$, where a crucial factor of $\frac{1}{n+1}$ appeared. In particular, for any ten-form $\phi_{(10)}$, not containing ρ's or $\mathrm{d}\rho$ we have

$$\int_{M_4 \times X} \phi_{(10)} \rho^n \mathrm{d}\rho = \sum_\alpha \frac{1}{n+1} \int_{M_4 \times Y_\alpha} \phi_{(10)} \ . \tag{10.4}$$

Using this function ρ we can now "cut off" the quantum fluctuations by simply defining

$$\widehat{C} := C\rho \quad , \quad \widehat{G} = G\rho \ . \tag{10.5}$$

The gauge invariant kinetic term of our theory is constructed from this field

$$S_{kin} = -\frac{1}{4\kappa_{11}^2} \int \widehat{G} \wedge \star \widehat{G} = -\frac{1}{4\kappa_{11}^2} \int_{r \ge R} \mathrm{d}C \wedge \star \mathrm{d}C \ . \tag{10.6}$$

However, the new field strength $\widehat{G}$ no longer is closed. This can be easily remedied by defining

$$\widetilde{C} := C\rho + B \wedge d\rho \quad , \quad \widetilde{G} := \mathrm{d}\widetilde{C} \tag{10.7}$$

Note that $\widetilde{G} = \widehat{G} + (C + \mathrm{d}B) \wedge \mathrm{d}\rho$, so we only modified $\hat{G}$ on the Y_α. The auxiliary field B living on Y_α has to be introduced in order to maintain gauge invariance of $\tilde{G}$. Its transformation law reads

$$\delta B = \Lambda \ , \tag{10.8}$$

which leads to

$$\delta\widetilde{C} = d(\Lambda\rho) \ . \tag{10.9}$$

Using these fields we are finally in a position to postulate the form of a modified topological term [P2],

$$\widetilde{S}_{top} \quad := \quad -\frac{1}{12\kappa_{11}^2}\int_{\mathbb{R}^4\times X}\widetilde{C}\wedge\widetilde{G}\wedge\widetilde{G}\ . \tag{10.10}$$

To see that this form is indeed useful for our purposes, one simple has to calculate its gauge transformation. After plugging in a Kaluza-Klein expansion of the fields, $C=\sum_i A^i\wedge\omega^i+\ldots$, $\Lambda=\sum_i\epsilon^i\omega^i+\ldots$, where the ω^i are harmonic two-forms on X (see [P2] and [P4] for details), we arrive at

$$\delta\widetilde{S}_{top}=-\sum_\alpha\frac{1}{(2\pi)^2 3!}\int_{\mathbb{R}^4}\epsilon^i F^j F^k\int_{Y_\alpha}(T_2)^3\omega^i\wedge\omega^j\wedge\omega^k\ . \tag{10.11}$$

Here we used $T_2:=\left(\frac{2\pi^2}{\kappa_{11}^2}\right)^{1/3}$. The quantity T_2 can be interpreted as the M2-brane tension [P3]. Note that the result is a sum of terms which are localized at Y_α. The corresponding Euclidean anomaly polynomial is given by

$$I_E^{(top)}=\sum_\alpha I_{E,\alpha}^{(top)}=-i\sum_\alpha I_{M,\alpha}^{(top)}=\sum_\alpha\frac{i}{(2\pi)^2 3!}F^iF^jF^k\int_{Y_\alpha}(T_2)^3\omega^i\wedge\omega^j\wedge\omega^k\ . \tag{10.12}$$

This is very similar to the gauge anomaly $I_\alpha^{(gauge)}$ and we do indeed get a local cancellation of the anomaly, provided we have

$$\int_{Y_\alpha}(T_2)^3\omega^i\wedge\omega^j\wedge\omega^k=\sum_{\sigma\in T_\alpha}q_\sigma^i q_\sigma^j q_\sigma^k\ . \tag{10.13}$$

(Note that the condition of local anomaly cancellation is $iI_\alpha^{(gauge)}+I_{E,\alpha}^{(top)}=0$, from (9.36).) In [152] it was shown that this equation holds for all known examples of conical singularities. It is particularly important that our modified topological term gives a sum of terms localized at Y_α without any integration by parts on X. This is crucial, because local quantities are no longer well-defined after an integration by parts[3].

After having seen how anomaly cancellation works in the case of gauge anomalies we turn to the mixed anomaly. In fact, it cannot be cancelled through an inflow mechanism from any of the terms in the action of eleven-dimensional supergravity. However, it was found in [131], [48] that there is a first correction term to the supergravity action, called the *Green-Schwarz term*. On a smooth manifold $\mathbb{R}^4\times X$ it reads

$$S_{GS}=-\frac{T_2}{2\pi}\int_{\mathbb{R}^4\times X}G\wedge X_7=-\frac{T_2}{2\pi}\int_{\mathbb{R}^4\times X}C\wedge X_8, \tag{10.14}$$

[3] Consider for example $\int_a^b df=f(b)-f(a)=(f(b)+c)-(f(a)+c)$. It is impossible to infer the value of f at the boundaries a and b.

with

$$X_8 := \frac{1}{(2\pi)^3 4!}\left(\frac{1}{8}\mathrm{tr}R^4 - \frac{1}{32}(\mathrm{tr}R^2)^2\right) . \tag{10.15}$$

and $X_8 = \mathrm{d}X_7$. The precise coefficient of the Green-Schwarz term was determined in [P3]. Then, there is a natural modification[4] on our singular manifold [P2],

$$\widetilde{S}_{GS} := -\frac{T_2}{2\pi}\int_{\mathbb{R}^4\times X} \widetilde{C} \wedge X_8 . \tag{10.16}$$

and its variation reads

$$\delta\widetilde{S}_{GS} = -\frac{T_2}{2\pi}\sum_\alpha \int_{\mathbb{R}^4\times Y_\alpha} \epsilon^i \omega^i \wedge X_8 . \tag{10.17}$$

To obtain this result we again used a Kaluza-Klein expansion, and we again did not integrate by parts. X_8 can be expressed in terms of the first and second Pontrjagin classes, $p_1 = -\frac{1}{2}\left(\frac{1}{2\pi}\right)^2 \mathrm{tr}R^2$ and $p_2 = \frac{1}{8}\left(\frac{1}{2\pi}\right)^4 [(\mathrm{tr}R^2)^2 - 2\mathrm{tr}R^4]$, as

$$X_8 = \frac{\pi}{4!}\left[\frac{p_1^2}{4} - p_2\right] . \tag{10.18}$$

The background we are working in is four-dimensional Minkowski space times a G_2-manifold. In this special setup the Pontrjagin classes can easily be expressed in terms of the Pontrjagin classes p_i' on $(\mathbb{R}^4, \eta)$ and those on (X, g), which we will write as p_i''. We have $p_1 = p_1' + p_1''$ and $p_2 = p_1' \wedge p_1''$. Using these relations we obtain a convenient expression for the inflow (10.17),

$$\delta\widetilde{S}_{GS} = \frac{T_2}{2\pi}\frac{\pi}{48}\sum_\alpha \int_{\mathbb{R}^4} \epsilon^i p_1' \int_{Y_\alpha} \omega^i \wedge p_1'' . \tag{10.19}$$

The corresponding (Euclidean) anomaly polynomial is given by

$$I_E^{(GS)} = \sum_\alpha I_{E,\alpha}^{(GS)} = -i\sum_\alpha \frac{1}{24}F^i p_1' \int_{Y_\alpha} \frac{T_2}{4}\omega^i \wedge p_1'' , \tag{10.20}$$

and we see that the mixed anomaly cancels locally provided

$$\int_{Y_\alpha} \frac{T_2}{4}\omega^i \wedge p_1'' = \sum_{\sigma\in T_\alpha} q_\sigma^i . \tag{10.21}$$

All known examples satisfy this requirement [152].

[4]The reader might object that in fact one could also use $\int \tilde{C}\wedge\tilde{X}_8$, $\int \tilde{G}\wedge X_8$ or $\int \tilde{G}\wedge\tilde{X}_8$. However, all these terms actually lead to the same result [P2].

10.2 Non-Abelian gauge groups and anomalies

Finally we also want to comment on anomaly cancellation in the case of non-Abelian gauge groups. The calculations are relatively involved and we refer the reader to [152] and [P2] for the details. We only present the basic mechanism. Non-Abelian gauge fields occur if X carries ADE singularities. The enhanced gauge symmetry can be understood to come from M2-branes that wrap the vanishing cycles in the singularity. Since ADE singularities have codimension four, the set of singular points is a three-dimensional submanifold Q of X. Chiral fermions which are charged under the non-Abelian gauge group are generated if the Q itself develops a conical singularity. Close to such a singularity P_α of Q the space X looks like a cone on some Y_α. If U_α denotes the intersection of Q with Y_α then, close to P_α, Q is a cone on U_α. In this case there are ADE gauge fields on $\mathbb{R}^4 \times Q$ which reduce to non-Abelian gauge fields on $\mathbb{R}^4$ if we perform a Kaluza-Klein expansion on Q. On the P_α we have a number of chiral multiplets Φ_σ which couple to both the non-Abelian gauge fields and the Abelian ones, coming from the Kaluza-Klein expansion of the C-field. Thus, we expect to get a $U(1)^3$, $U(1)H^2$ and H^3 anomaly[5], where H is the relevant ADE gauge group. The relevant anomaly polynomial for this case is (again taking into account the negative chirality of the fermions)

$$I_6 = -2\pi[(-1)\hat{A}(M)\mathrm{ch}(F^{(Ab)})\mathrm{ch}(F)]_6 \tag{10.22}$$

where $F^{(Ab)} := iq^i F^i$ denotes the Abelian and F the non-Abelian gauge field. Expansion of this formula gives four terms namely (10.2) and

$$I^{(H^3)} = -\frac{i}{(2\pi)^2 3!}\mathrm{tr}F^3 \quad , \quad I^{(U(1)_i H^2)} = \frac{1}{(2\pi)^2 2} q_i F_i \mathrm{tr}F^2 \ . \tag{10.23}$$

It turns out [152], [P2] that our special setup gives rise to two terms on $\mathbb{R}^4 \times Q$, which read

$$\widetilde{S}_1 = -\frac{i}{6(2\pi)^2}\int_{\mathbb{R}^4\times Q} K \wedge \mathrm{tr}(\widetilde{A}\widetilde{F}^2) \ , \tag{10.24}$$

$$\widetilde{S}_2 = \frac{T_2}{2(2\pi)^2}\int_{\mathbb{R}^4\times Q} \widetilde{C} \wedge \mathrm{tr}\widetilde{F}^2 \ . \tag{10.25}$$

Here K is the curvature of a certain line bundle described in [152], and $\widetilde{A}$ and $\widetilde{F}$ are modified versions of A and F, the gauge potential and field strength of the non-Abelian ADE gauge field living on $\mathbb{R}^4 \times Q$. The variation of these terms leads to contributions which are localised at the various conical singularities, and after continuation to Euclidean space the corresponding polynomials cancel the anomalies (10.23) locally, i.e. separately at each conical singularity. For the details of this mechanism the reader is referred to [P2]. The main steps are similar to what we did in the last chapter, and the only difficulty comes from the non-Abelian nature of the fields which complicates the calculation.

[5] The $U(1)^2G$ anomaly is not present as tr T_a vanishes for all generators of ADE gauge groups, and the H^3-anomaly is only present for $H = SU(n)$.

Chapter 11

Compact Weak G_2-Manifolds

We have seen already that one possible vacuum of eleven-dimensional supergravity is given by the direct product of AdS_4 with a compact Einstein seven-manifold of positive curvature, together with a flux $G_{\mu\nu\rho\sigma} \propto \epsilon_{\mu\nu\rho\sigma}$. Furthermore, if the compact space carries exactly one Killing spinor we are left with $\mathcal{N} = 1$ in four dimensions. Such manifolds are known as weak G_2-manifolds. As for the case of G_2-manifolds one expects charged chiral fermions to occur, if the compact manifold carries conical singularities. Unfortunately, no explicit metric for a compact G_2-manifold with conical singularities is known. However, in [P1] explicit metrics for compact weak G_2-manifolds with conical singularities have been constructed. These spaces are expected to share many properties of singular compact G_2-manifolds, and are therefore useful to understand the structure of the latter.

The strategy to construct the compact weak G_2-holonomy manifolds is the following: we begin with any non-compact G_2-holonomy manifold X that asymptotically, for "large r" becomes a cone on some 6-manifold Y. Manifolds of this type have been constructed in [63]. The G_2-holonomy of X implies certain properties of the 6-manifold Y which we deduce. In fact, Y can be any Einstein space of positive curvature with weak $SU(3)$-holonomy. Then we use this Y to construct a compact weak G_2-holonomy manifold X_λ with two conical singularities that, close to the singularities, looks like a cone on Y.

11.1 Properties of weak G_2-manifolds

On a weak G_2-manifold there exists a unique Killing spinor[1],

$$\left(\partial_j + \frac{1}{4}\omega_j^{ab}\gamma^{ab}\right)\theta = i\frac{\lambda}{2}\gamma_j\theta \ , \tag{11.1}$$

from which one can construct a three-form

$$\Phi_\lambda := \frac{i}{6}\theta^\tau\gamma_{abc}\theta \ e^a \wedge e^b \wedge e^c \ , \tag{11.2}$$

[1] Note that a, b, c are "flat" indices with Euclidean signature, and upper and lower indices are equivalent.

which satisfies

$$d\Phi_\lambda = 4\lambda\, {}^*\Phi_\lambda \ . \tag{11.3}$$

Furthermore, (11.1) implies that X_λ has to be Einstein,

$$\mathcal{R}_{ij} = 6\lambda^2 g_{ij} \ . \tag{11.4}$$

It can be shown that the converse statement is also true, namely that Eq. (11.3) implies the existence of a spinor satisfying (11.1). Note that for $\lambda \to 0$, at least formally, weak G_2 goes over to G_2-holonomy.

To proceed we define the quantity ψ_{abc} to be totally antisymmetric with $\psi_{123} = \psi_{516} = \psi_{624} = \psi_{435} = \psi_{471} = \psi_{673} = \psi_{572} = 1$, and its dual

$$\hat{\psi}_{abcd} := \frac{1}{3!}\tilde{\epsilon}^{abcdefg}\psi_{efg} \ . \tag{11.5}$$

Using these quantities we note that every antisymmetric tensor A^{ab} transforming as the **21** of SO(7) can always be decomposed [23] into a piece A^{ab}_+ transforming as the **14** of G_2 (called self-dual) and a piece A^{ab}_- transforming as the **7** of G_2 (called anti-self-dual):

$$\begin{aligned} A^{ab} &= A^{ab}_+ + A^{ab}_- && (11.6)\\ A^{ab}_+ &= \frac{2}{3}\left(A^{ab} + \frac{1}{4}\hat{\psi}^{abcd}A^{cd}\right) =: \mathcal{P}_{\mathbf{14}}A^{ab} \ , && (11.7)\\ A^{ab}_- &= \frac{1}{3}\left(A^{ab} - \frac{1}{2}\hat{\psi}^{abcd}A^{cd}\right) =: \mathcal{P}_{\mathbf{7}}A^{ab} \ , && (11.8)\end{aligned}$$

with orthogonal projectors $(\mathcal{P}_{\mathbf{14}})^{cd}_{ab} := \frac{2}{3}\left(\delta^{cd}_{ab} + \frac{1}{4}\hat{\psi}_{ab}{}^{cd}\right)$ and $(\mathcal{P}_{\mathbf{7}})^{cd}_{ab} := \frac{1}{3}\left(\delta^{cd}_{ab} - \frac{1}{2}\hat{\psi}_{ab}{}^{cd}\right)$, $\delta^{cd}_{ab} := \frac{1}{2}(\delta^c_a\delta^d_b - \delta^d_a\delta^c_b)$. Using the identity[2]

$$\hat{\psi}_{abde}\psi_{dec} = -4\psi_{abc} \tag{11.9}$$

we find that the self-dual[3] part satisfies

$$\psi^{abc}A^{bc}_+ = 0 \ . \tag{11.11}$$

In particular, one has

$$\omega^{ab}\gamma^{ab} = \omega^{ab}_+\gamma^{ab}_+ + \omega^{ab}_-\gamma^{ab}_- \ . \tag{11.12}$$

The importance of (anti-)self-duality will become clear in the following theorem.

[2] Many useful identities of this type are listed in the appendix of [23].

[3] The reader might wonder how this name is motivated, since so far we did not encounter a self-duality condition. As a matter of fact one can show that for any anti-symmetric tensor B^{ab} the following three statements are equivalent

$$\begin{aligned} \psi_{abc}B^{bc} &= 0 \ ,\\ B^{ab}_- &= 0 \ ,\\ B^{ab} &= \frac{1}{2}\hat{\psi}_{abcd}B^{cd} \ , \end{aligned} \tag{11.10}$$

and the last equation now explains the nomenclature.

Theorem 11.1 *A manifold (M, g) is a (weak) G_2-manifold, if and only if there exists a frame in which the spin connection satisfies*

$$\psi_{abc}\omega^{bc} = -2\lambda\, e^a \ , \tag{11.13}$$

where e^a is the 7-bein on X.

We will call such a frame a *self-dual frame.* The proof can be found in [23]. In the case of a G_2-manifold one simply has to take $\lambda = 0$. We also need the following:

Proposition 11.2 *The three-form Φ_λ of Eq.(11.2) can be written as*

$$\Phi_\lambda = \frac{1}{6}\psi_{abc}\ e^a \wedge e^b \wedge e^c \tag{11.14}$$

if and only if the 7-beins e^a are a self-dual frame. This holds for $\lambda = 0$ (G_2) and $\lambda \neq 0$ (weak G_2).

Later, for weak G_2, we will consider a frame which is not self-dual and thus the 3-form Φ_λ will be slightly more complicated than (11.14). To prove the proposition it will be useful to have an explicit representation for the γ-matrices in 7 dimensions. A convenient representation is in terms of the ψ_{abc} as [23]

$$(\gamma_a)_{AB} = i(\psi_{aAB} + \delta_{aA}\delta_{8B} - \delta_{aB}\delta_{8A}) \ . \tag{11.15}$$

Here $a = 1, \dots 7$ while $A, B = 1, \dots 8$ and it is understood that $\psi_{aAB} = 0$ if A or B equals 8. One then has [23]

$$\begin{aligned}
(\gamma_{ab})_{AB} &= \hat{\psi}_{abAB} + \psi_{abA}\delta_{8B} - \psi_{abB}\delta_{8A} + \delta_{aA}\delta_{bB} - \delta_{aB}\delta_{bA} \ , && (11.16)\\
(\gamma_{abc})_{AB} &= i\psi_{abc}(\delta_{AB} - 2\delta_{8A}\delta_{8B}) - 3i\psi_{A[ab}\delta_{c]B} - 3i\psi_{B[ab}\delta_{c]A} \\
&\quad -i\hat{\psi}_{abcA}\delta_{8B} - i\hat{\psi}_{abcB}\delta_{8A} \ . && (11.17)
\end{aligned}$$

In order to see under which condition (11.2) reduces to (11.14) we use the explicit representation for the γ-matrices (11.15) given above. It is then easy to see that $\theta^T\gamma_{abc}\theta \sim \psi_{abc}$ if and only if $\theta_A \sim \delta_{8A}$. This means that our 3-form Φ is given by (11.14) if and only if the covariantly constant, resp. Killing spinor θ only has an eighth component, which then must be a constant which we can take to be 1. With this normalisation we have

$$\theta^T\gamma_{abc}\theta = -i\psi_{abc} \ , \tag{11.18}$$

so that Φ is correctly given by (11.14). From the above explicit expression for γ_{ab} one then deduces that $(\gamma_{ab})_{AB}\theta_B = \psi_{abA}$ and $\omega^{ab}(\gamma_{ab})_{AB}\theta_B = \omega^{ab}\psi_{abA}$. Also, $i(\gamma_c)_{AB}\theta_B = -\delta_{cA}$, so that Eq. (11.1) reduces to $\omega^{ab}\psi_{abc} = -2\lambda e^c$.

11.2 Construction of weak G_2-holonomy manifolds with singularities

Following [P1] we start with any (non-compact) G_2-manifold X which asymptotically is a cone on a compact 6-manifold Y,

$$\mathrm{d}s_X^2 \sim \mathrm{d}r^2 + r^2 \mathrm{d}s_Y^2 \ . \tag{11.19}$$

Since X is Ricci flat, Y must be an Einstein manifold with $\mathcal{R}_{\alpha\beta} = 5\delta_{\alpha\beta}$. In practice [63], $Y = \mathbb{CP}^3$, $S^3 \times S^3$ or $SU(3)/U(1)^2$, with explicitly known metrics. On Y we introduce 6-beins

$$\mathrm{d}s_Y^2 = \sum_{\alpha=1}^{6} \widetilde{\epsilon}^{\alpha} \otimes \widetilde{\epsilon}^{\alpha} \ , \tag{11.20}$$

and similarly on X

$$\mathrm{d}s_X^2 = \sum_{a=1}^{7} \hat{e}^{a} \otimes \hat{e}^{a} \ . \tag{11.21}$$

Our conventions are that $a, b, \ldots$ run from 1 to 7 and $\alpha, \beta, \ldots$ from 1 to 6. The various manifolds and corresponding viel-beins are summarized in the table below. Since X has G_2-holonomy we may assume that the 7-beins $\hat{e}^a$ are chosen such that the ω^{ab} are self-dual, and hence we know from the above remark that the closed and co-closed 3-form Φ is simply given by Eq. (11.14), i.e. $\Phi = \frac{1}{6}\psi_{abc}\, \hat{e}^a \wedge \hat{e}^b \wedge \hat{e}^c$. Although there is such a self-dual choice, in general, we are not guaranteed that this choice is compatible with the natural choice of 7-beins on X consistent with a cohomogeneity-one metric as (11.19). (For any of the three examples cited above, the self-dual choice actually is compatible with a cohomogeneity-one metric.)

X	Y	X_c	X_λ
$\hat{e}^a$	$\widetilde{\epsilon}^\alpha$	$\overline{e}^a$	e^a
Φ		ϕ	Φ_λ

Table 1 : The various manifolds, corresponding viel-beins and 3-forms that enter our construction.

Now we take the limit $X \to X_c$ in which the G_2-manifold becomes exactly a cone on Y so that $\hat{e}^a \to \overline{e}^a$ with

$$\overline{e}^\alpha = r\widetilde{\epsilon}^\alpha \quad , \quad \overline{e}^7 = \mathrm{d}r \ . \tag{11.22}$$

In this limit the cohomogeneity-one metric can be shown to be compatible with the self-dual choice of frame (see [P1] for a proof) so that we may assume that (11.22) is such a self-dual frame. More precisely, we may assume that the original frame $\hat{e}^a$ on X was chosen in such a way that after taking the conical limit the $\overline{e}^a$ are a self-dual

frame. Then we know that the 3-form Φ of X becomes a 3-form ϕ of X_c given by the limit of (11.14), namely

$$\phi = r^2 \mathrm{d}r \wedge \xi + r^3 \zeta \ , \tag{11.23}$$

with the 2- and 3-forms on Y defined by

$$\xi = \frac{1}{2}\psi_{7\alpha\beta}\, \widetilde{\epsilon}^{\alpha} \wedge \widetilde{\epsilon}^{\beta} \quad , \quad \zeta = \frac{1}{6}\psi_{\alpha\beta\gamma}\, \widetilde{\epsilon}^{\alpha} \wedge \widetilde{\epsilon}^{\beta} \wedge \widetilde{\epsilon}^{\gamma} \ . \tag{11.24}$$

The dual 4-form is given by

$$^*\phi = r^4\, {}^{*_Y}\xi - r^3 \mathrm{d}r \wedge {}^{*_Y}\zeta \ , \tag{11.25}$$

where ${}^{*_Y}\xi$ is the dual of ξ in Y.[4] As for the original Φ, after taking the conical limit, we still have $\mathrm{d}\phi = 0$ and $\mathrm{d}^*\phi = 0$. This is equivalent to

$$\begin{aligned} \mathrm{d}\xi &= 3\zeta \ , \\ \mathrm{d}\, {}^{*_Y}\zeta &= -4\, {}^{*_Y}\xi \ . \end{aligned} \tag{11.30}$$

These are properties of appropriate forms on Y, and they can be checked to be true for any of the three standard Y's. Actually, these relations show that Y has weak $SU(3)$-holonomy. Conversely, if Y is a 6-dimensional manifold with weak $SU(3)$-holonomy, then we know that these forms exist. This is analogous to the existence of the 3-form Φ_λ with $\mathrm{d}\Phi_\lambda = 4\lambda^*\Phi_\lambda$ for weak G_2-holonomy. These issues were discussed e.g. in [79]. Combining the two relations (11.30), we see that on Y there exists a 2-form ξ obeying

$$\mathrm{d}\, {}^{*_Y}\mathrm{d}\xi + 12\, {}^{*_Y}\xi = 0 \quad , \quad \mathrm{d}\, {}^{*_Y}\xi = 0 \ . \tag{11.31}$$

[4] We need to relate Hodge duals on the 7-manifolds X, X_c or X_λ to the Hodge duals on the 6-manifold Y. To do this, we do not need to specify the 7-manifold and just call it X_7. We assume that the 7-beins of X_7, called e^a, and the 6-beins of Y, called $\widetilde{\epsilon}^\alpha$ can be related by

$$e^7 = \mathrm{d}r \ , \quad e^\alpha = h(r)\widetilde{\epsilon}^\alpha \ . \tag{11.26}$$

We denote the Hodge dual of a form π on X_7 simply by $^*\pi$ while the 6-dimensional Hodge dual of a form σ on Y is denoted ${}^{*_Y}\sigma$. The duals of p-forms on X_7 and on Y are defined in terms of their respective viel-bein basis, namely

$$^*\left(e^{a_1} \wedge \ldots \wedge e^{a_p}\right) = \frac{1}{(7-p)!}\widetilde{\epsilon}^{a_1 \ldots a_p}{}_{b_1 \ldots b_{7-p}} e^{b_1} \wedge \ldots \wedge e^{b_{7-p}} \tag{11.27}$$

and

$${}^{*_Y}\left(\widetilde{\epsilon}^{\alpha_1} \wedge \ldots \wedge \widetilde{\epsilon}^{\alpha_p}\right) = \frac{1}{(6-p)!}\widetilde{\epsilon}^{\alpha_1 \ldots \alpha_p}{}_{\beta_1 \ldots \beta_{6-p}} \widetilde{\epsilon}^{\beta_1} \wedge \ldots \wedge \widetilde{\epsilon}^{\beta_{6-p}} \ . \tag{11.28}$$

Here the $\widetilde{\epsilon}$-tensors are the "flat" ones that equal ± 1. Expressing the e^a in terms of the $\widetilde{\epsilon}^\alpha$ provides the desired relations. In particular, for a p-form ω_p on Y we have

$$\begin{aligned} ^*\left(\mathrm{d}r \wedge \omega_p\right) &= h(r)^{6-2p}\, {}^{*_Y}\omega_p \ , \\ ^*\omega_p &= (-)^p h(r)^{6-2p} \mathrm{d}r \wedge {}^{*_Y}\omega_p \ , \end{aligned} \tag{11.29}$$

where we denote both the form on Y and its trivial extension onto X_λ by the same symbol ω_p.

This implies $\Delta_Y \xi = 12\xi$, where $\Delta_Y = -{}^{*_Y}\mathrm{d}\,{}^{*_Y}\mathrm{d} - \mathrm{d}\,{}^{*_Y}\mathrm{d}\,{}^{*_Y}$ is the Laplace operator on forms on Y. Note that with $\zeta = \frac{1}{3}\mathrm{d}\xi$ we actually have $\phi = \mathrm{d}\left(\frac{r^3}{3}\xi\right)$ and ϕ is cohomologically trivial. This was not the case for the original Φ.

We now construct a manifold X_λ with a 3-form Φ_λ that is a deformation of this 3-form ϕ and that will satisfy the condition (11.3) for weak G_2-holonomy. Since weak G_2-manifolds are Einstein manifolds we need to introduce some scale r_0 and make the following ansatz for the metric on X_λ

$$\mathrm{d}s^2_{X_\lambda} = \mathrm{d}r^2 + r_0^2 \sin^2 \hat{r}\ \mathrm{d}s^2_Y \ , \tag{11.32}$$

with

$$\hat{r} = \frac{r}{r_0} \quad , \quad 0 \le r \le \pi r_0 \ . \tag{11.33}$$

Clearly, this metric has two conical singularities, one at $r = 0$ and the other at $r = \pi r_0$.

We see from (11.32) that we can choose 7-beins e^a on X_λ that are expressed in terms of the 6-beins $\widetilde{\epsilon}^\alpha$ of Y as

$$e^\alpha = r_0 \sin \hat{r}\ \widetilde{\epsilon}^\alpha \quad , \quad e^7 = \mathrm{d}r \ . \tag{11.34}$$

Although this is the natural choice, it should be noted that it is *not* the one that leads to a self-dual spin connection ω^{ab} that satisfies Eq. (11.13). We know from [23] that such a self-dual choice of 7-beins must exist if the metric (11.32) has weak G_2-holonomy but, as noted earlier, there is no reason why this choice should be compatible with cohomogeneity-one, i.e choosing $e^7 = \mathrm{d}r$. Actually, it is easy to see that for weak G_2-holonomy, $\lambda \neq 0$, a cohomogeneity-one choice of frame and self-duality are incompatible: a cohomogeneity-one choice of frame means $e^7 = \mathrm{d}r$ and $e^\alpha = h_{(\alpha)}(r)\widetilde{\epsilon}^\alpha$ so that $\omega^{\alpha\beta} = \frac{h_{(\alpha)}(r)}{h_{(\beta)}(r)}\widetilde{\omega}^{\alpha\beta}$ and $\omega^{\alpha 7} = h'_{(\alpha)}(r)\widetilde{\epsilon}^\alpha$. But then the self-duality condition for $a = 7$ reads $\psi_{7\alpha\beta}\frac{h_{(\alpha)}(r)}{h_{(\beta)}(r)}\widetilde{\omega}^{\alpha\beta} = -2\lambda \mathrm{d}r$. Since $\widetilde{\omega}^{\alpha\beta}$ is the spin connection on Y, associated with $\widetilde{\epsilon}^\alpha$, it contains no $\mathrm{d}r$-piece, and the self-duality condition cannot hold unless $\lambda = 0$.

Having defined the 7-beins on X_λ in terms of the 6-beins on Y, the Hodge duals on X_λ and on Y are related accordingly. If ω_p is a p-form on Y, we have

$$\begin{aligned} {}^*\left(\mathrm{d}r \wedge \omega_p\right) &= \left(r_0 \sin \hat{r}\right)^{6-2p}\ {}^{*_Y}\omega_p \\ {}^*\omega_p &= (-)^p \left(r_0 \sin \hat{r}\right)^{6-2p} \mathrm{d}r \wedge {}^{*_Y}\omega_p \ , \end{aligned} \tag{11.35}$$

where we denote both the form on Y and its trivial (r-independent) extension onto X_λ by the same symbol ω_p.

Finally, we are ready to determine the 3-form Φ_λ satisfying $\mathrm{d}\Phi_\lambda = \lambda^*\Phi_\lambda$. We make the ansatz [P1]

$$\Phi_\lambda = \left(r_0 \sin \hat{r}\right)^2 \mathrm{d}r \wedge \xi + \left(r_0 \sin \hat{r}\right)^3 \left(\cos \hat{r}\ \zeta + \sin \hat{r}\ \rho\right) \ . \tag{11.36}$$

Here, the 2-form ξ and the 3-forms ζ and ρ are forms on Y which are trivially extended to forms on X_λ (no r-dependence). Note that this Φ_λ is not of the form (11.14) as

the last term is not just ζ but $\cos\hat{r}\ \zeta + \sin\hat{r}\ \rho$. This was to be expected since the cohomogeneity-one frame cannot be self-dual. The Hodge dual of Φ_λ then is given by

$$ {}^*\Phi_\lambda = (r_0 \sin\hat{r})^4\, {}^{*Y}\xi - (r_0\sin\hat{r})^3\, \mathrm{d}r \wedge (\cos\hat{r}\, {}^{*Y}\zeta + \sin\hat{r}\, {}^{*Y}\rho) \tag{11.37} $$

while

$$ \begin{aligned} \mathrm{d}\Phi_\lambda &= (r_0\sin\hat{r})^2\, \mathrm{d}r \wedge (-\mathrm{d}\xi + 3\zeta) + (r_0 \sin\hat{r})^3\ (\cos\hat{r}\ \mathrm{d}\zeta + \sin\hat{r}\ \mathrm{d}\rho) \\ &+ \frac{4}{r_0}(r_0\sin\hat{r})^3\, \mathrm{d}r \wedge (\cos\hat{r}\ \rho - \sin\hat{r}\ \zeta)\ . \end{aligned} \tag{11.38} $$

In the last term, the derivative ∂_r has exchanged $\cos\hat{r}$ and $\sin\hat{r}$ and this is the reason why both of them had to be present in the first place.

Requiring $\mathrm{d}\Phi_\lambda = 4\lambda^*\Phi_\lambda$ leads to the following conditions

$$ \mathrm{d}\xi = 3\zeta\ , \tag{11.39} $$
$$ \mathrm{d}\rho = 4\lambda r_0\, {}^{*Y}\xi\ , \tag{11.40} $$
$$ \rho = -\lambda r_0\, {}^{*Y}\zeta\ , \tag{11.41} $$
$$ \zeta = \lambda r_0\, {}^{*Y}\rho\ . \tag{11.42} $$

Equations (11.41) and (11.42) require

$$ r_0 = \frac{1}{\lambda} \tag{11.43} $$

and $\zeta = {}^{*Y}\rho \ \Leftrightarrow\ \rho = -\,{}^{*Y}\zeta$ (since for a 3-form ${}^{*Y}({}^{*Y}\omega_3) = -\omega_3$). Then (11.40) is $\mathrm{d}\rho = 4\,{}^{*Y}\xi$, and inserting $\rho = -\,{}^{*Y}\zeta$ and Eq. (11.39) we get

$$ \mathrm{d}\,{}^{*Y}\mathrm{d}\xi + 12\,{}^{*Y}\xi = 0 \quad \text{and} \quad \mathrm{d}\,{}^{*Y}\xi = 0\ . \tag{11.44} $$

But we know from (11.31) that there is such a two-form ξ on Y. Then pick such a ξ and let $\zeta = \frac{1}{3}\mathrm{d}\xi$ and $\rho = -\,{}^{*Y}\zeta = -\frac{1}{3}\,{}^{*Y}\mathrm{d}\xi$. We conclude that

$$ \Phi_\lambda = \left(\frac{\sin\lambda r}{\lambda}\right)^2 \mathrm{d}r\wedge\xi + \frac{1}{3}\left(\frac{\sin\lambda r}{\lambda}\right)^3 (\cos\lambda r\, \mathrm{d}\xi - \sin\lambda r\ {}^{*Y}\mathrm{d}\xi) \tag{11.45} $$

satisfies $\mathrm{d}\Phi_\lambda = 4\lambda^*\Phi_\lambda$ and that the manifold with metric (11.32) has weak G_2-holonomy. Thus we have succeeded to construct, for every non-compact G_2-manifold that is asymptotically (for large r) a cone on Y, a corresponding compact weak G_2-manifold X_λ with two conical singularities that look, for small r, like cones on the same Y. Of course, one could start directly with any 6-manifold Y of weak $SU(3)$-holonomy.

The quantity λ sets the scale of the weak G_2-manifold X_λ which has a size of order $\frac{1}{\lambda}$. As $\lambda \to 0$, X_λ blows up and, within any fixed finite distance from $r = 0$, it looks like the cone on Y we started with.

Cohomology of the weak G_2-manifolds

As mentioned above, what one wants to do with our weak G_2-manifold at the end of the day is to compactify M-theory on it, and generate an interesting four-dimensional effective action. This is done by a Kaluza-Klein compactification (reviewed for instance in [P4]), and it is therefore desirable to know the cohomology groups of the compact manifold. To be more precise, there are various ways to define harmonic forms which are all equivalent on a compact manifold without singularities where one can freely integrate by parts. Since X_λ has singularities we must be more precise about the definition we adopt and about the required behaviour of the forms as the singularities are approached.

Physically, when one does a Kaluza-Klein reduction of an eleven-dimensional k-form C_k one first writes a double expansion $C_k = \sum_{p=0}^{k} \sum_i A^i_{k-p} \wedge \phi^i_p$ where A^i_{k-p} are $(k-p)$-form fields in four dimensions and the ϕ^i_p constitute, for each p, a basis of p-form fields on X_λ. It is convenient to expand with respect to a basis of eigenforms of the Laplace operator on X_λ. Indeed, the standard kinetic term for C_k becomes

$$\begin{aligned}\int_{\mathcal{M}_4 \times X_\lambda} \mathrm{d}C_k \wedge {}^*\mathrm{d}C_k \;&=\; \sum_{p,i} \left(\int_{\mathcal{M}_4} \mathrm{d}A^i_{k-p} \wedge {}^*\mathrm{d}A^i_{k-p} \;\int_{X_\lambda} \phi^i_p \wedge {}^*\phi^i_p \right.\\ &\quad + \left. \int_{\mathcal{M}_4} A^i_{k-p} \wedge {}^*A^i_{k-p} \;\int_{X_\lambda} \mathrm{d}\phi^i_p \wedge {}^*\mathrm{d}\phi^i_p \right) . \end{aligned} \tag{11.46}$$

Then a massless field A_{k-p} in four dimensions arises for every closed p-form ϕ^i_p on X_λ for which $\int_{X_\lambda} \phi_p \wedge {}^*\phi_p$ is finite. Moreover, the usual gauge condition $\mathrm{d}^*C_k = 0$ leads to the analogous four-dimensional condition $\mathrm{d}^*A_{k-p} = 0$ provided we also have $\mathrm{d}^*\phi_p = 0$. We are led to the following definition:

Definition 11.3 An L^2-harmonic p-form ϕ_p on X_λ is a p-form such that

$$(i) \qquad ||\phi_p||^2 \equiv \int_{X_\lambda} \phi_p \wedge {}^*\phi_p < \infty \ , \quad \text{and} \tag{11.47}$$

$$(ii) \qquad \mathrm{d}\phi_p = 0 \quad \text{and} \quad \mathrm{d}^*\phi_p = 0 \ . \tag{11.48}$$

Then one can prove [P1] the following:

Proposition 11.4 *Let X_λ be a 7-dimensional manifold with metric given by (11.32), (11.33). Then all L^2-harmonic p-forms ϕ_p on X_λ for $p \leq 3$ are given by the trivial (r-independent) extensions to X_λ of the L^2-harmonic p-forms ω_p on Y. For $p \geq 4$ all L^2-harmonic p-forms on X_λ are given by ${}^*\phi_{7-p}$.*

Since there are no harmonic 1-forms on Y we immediately have the
Corollary : The Betti numbers on X_λ are given by those of Y as

$$\begin{aligned} b^0(X_\lambda) = b^7(X_\lambda) = 1 \quad &, \quad b^1(X_\lambda) = b^6(X_\lambda) = 0 \ , \\ b^2(X_\lambda) = b^5(X_\lambda) = b^2(Y) \quad &, \quad b^3(X_\lambda) = b^4(X_\lambda) = b^3(Y) \ . \end{aligned} \tag{11.49}$$

The proof of the proposition is lengthy and rather technical and the reader is referred to [P1] for details.

Chapter 12

The Hořava-Witten Construction

The low-energy effective theory of M-theory is eleven-dimensional supergravity. Over the last years various duality relations involving string theories and eleven-dimensional supergravity have been established, confirming the evidence for a single underlying theory. One of the conjectured dualities, discovered by Hořava and Witten [83], relates M-theory on the orbifold $M_{10} \times S^1/\mathbb{Z}_2$ to $E_8 \times E_8$ heterotic string theory on the manifold M_{10}. In [83] it was shown that the gravitino field ψ_M, $M, N, \ldots = 0, 1, \ldots 10$, present in the eleven-dimensional bulk $M_{10} \times S^1/\mathbb{Z}_2$ leads to an anomaly on the ten-dimensional fixed "planes" of this orbifold. Part of this anomaly can be cancelled if we introduce a ten-dimensional E_8 vector multiplet on each of the two fixed planes. This does not yet cancel the anomaly completely. However, once the vector multiplets are introduced they have to be coupled to the eleven-dimensional bulk theory. In [83] it was shown that this leads to a modification of the Bianchi identity to $\mathrm{d}G \neq 0$, which in turn leads to yet another contribution to the anomaly, coming from the non-invariance of the classical action. Summing up all these terms leaves us with an anomaly free theory. However, the precise way of how all these anomalies cancel has been the subject of quite some discussion in the literature. Using methods similar to the ones we described in chapter 10, and building on the results of [22], we were able to prove for the first time [P3] that the anomalies do actually cancel locally, i.e. separately on each of the two fixed planes. In this chapter we will explain the detailed mechanism that leads to this local anomaly cancellation.

The orbifold $\mathbb{R}^{10} \times S^1/\mathbb{Z}_2$

Let the eleven-dimensional manifold M_{11} be the Riemannian product of ten-manifold (M_{10}, g) and a circle S^1 with its standard metric. The coordinates on the circle are taken to be $\phi \in [-\pi, \pi]$ with the two endpoints identified. In particular, the radius of the circle will be taken to be one. The equivalence classes in $S^1/\mathbb{Z}_2$ are the pairs of points with coordinate ϕ and $-\phi$, i.e. $\mathbb{Z}_2$ acts as $\phi \to -\phi$. This map has the fixed points 0 and π, thus the space $M_{10} \times S^1/\mathbb{Z}_2$ contains two singular ten-dimensional spaces. In the simplest case we have $M_{10} = \mathbb{R}^{10}$, which is why we call these spaces fixed planes.

Before we proceed let us introduce some nomenclature. Working on the space $M_{10}\times S^1$, with an additional $\mathbb{Z}_2$-projection imposed, is called to work in the "upstairs" formalism. Equivalently, one might work on the manifold $M_{10}\times I \cong M_{10}\times S^1/\mathbb{Z}_2$ with $I=[0,\pi]$. This is referred to as the "downstairs" approach. It is quite intuitive to work downstairs on the interval but for calculational purposes it is more convenient to work on manifolds without boundary. Otherwise one would have to impose boundary conditions for the fields. Starting from supergravity on $M_{10}\times I$ it is easy to obtain the action in the upstairs formalism. One simply has to use $\int_I \ldots = \frac{1}{2}\int_{S^1}\ldots$.

Working upstairs one has to impose the $\mathbb{Z}_2$-projection by hand. By inspection of the topological term of eleven-dimensional supergravity (c.f. Eq. (9.3)) one finds that $C_{\mu\nu\rho}$, with $\mu,\nu,\ldots=0,1,\ldots 9$, is $\mathbb{Z}_2$-odd, whereas $C_{\mu\nu 10}$ is $\mathbb{Z}_2$-even. This implies that $C_{\mu\nu\rho}$ is projected out and C can be written as $C=B\wedge d\phi$.

Following [22] we define for further convenience

$$\begin{aligned} \delta_1 &:= \delta(\phi)d\phi \quad , \quad \delta_2 := \delta(\phi-\pi)d\phi \ , \\ \epsilon_1(\phi) &:= \mathrm{sig}(\phi)-\frac{\phi}{\pi} \quad , \quad \epsilon_2(\phi) := \epsilon_1(\phi-\pi) \ , \end{aligned} \tag{12.1}$$

which are well-defined on S^1 and satisfy

$$d\epsilon_i = 2\delta_i - \frac{d\phi}{\pi} \ . \tag{12.2}$$

After regularization we get [22] [1]

$$\delta_i\epsilon_j\epsilon_k \to \frac{1}{3}(\delta_{ji}\delta_{ki})\delta_i \ . \tag{12.3}$$

Anomalies of M-theory on $M_{10}\times S^1/\mathbb{Z}_2$

Next we need to study the field content of eleven-dimensional supergravity on the given orbifold, and analyse the corresponding anomalies. Compactifying eleven-dimensional supergravity on the circle leads to a set of (ten-dimensional) massless fields which are independent of the coordinate ϕ and other (ten-dimensional) massive modes. Only the former can lead to anomalies in ten-dimensions. For instance, the eleven-dimensional Rarita-Schwinger field reduces to a sum of infinitely many massive modes, and two massless ten-dimensional gravitinos of opposite chirality. On our orbifold we have to impose the $\mathbb{Z}_2$-projection on these fields. Only one of the two ten-dimensional gravitinos is even under $\phi\to-\phi$, and the other one is projected out. Therefore, after the projection we are left with a chiral theory, which, in general, is anomalous. Note that we have not taken the radius of the compactification to zero, and there is a ten-plane for every point in $S^1/\mathbb{Z}_2$. Therefore it is not clear a priori on which ten-dimensional plane the anomalies should occur. There are, however, two very special ten-planes, namely those fixed by the $\mathbb{Z}_2$-projection. It is therefore natural to assume that the

[1] Of course δ_{ij} is the usual Kronecker symbol, not to be confused with δ_i.

anomalies should localise on these planes. Clearly, the entire setup is symmetric and the two fixed planes should carry the same anomaly. In the case in which the radius of S^1 reduces to zero we get the usual ten-dimensional anomaly of a massless gravitino field. We conclude that in the case of finite radius the anomaly which is situated on the ten-dimensional planes is given by exactly one half of the usual gravitational anomaly in ten dimensions. This result has first been derived in [83]. Since the $\mathbb{Z}_2$-projection gives a positive chirality spin-$\frac{3}{2}$ field and a negative chirality spin-$\frac{3}{2}$ field in the Minkowskian, (recall $\Gamma^E = -\Gamma^M$), we are led to a negative chirality spin-$\frac{3}{2}$ and a positive chirality spin-$\frac{1}{2}$ field in the Euclidean. The anomaly polynomial of these so-called untwisted fields on a single fixed plane reads

$$I_{12(i)}^{(untwisted)} = \frac{1}{2}\left\{\frac{1}{2}\left(-I_{grav}^{(3/2)}(R_i) + I_{grav}^{(1/2)}(R_i)\right)\right\} . \tag{12.4}$$

The second factor of $\frac{1}{2}$ arises because the fermions are Majorana-Weyl. $i = 1, 2$ denotes the two planes, and R_i is the curvature two-form on the i-th plane. $I_{grav}^{(3/2)}$ is obtained from $I_{12}^{(3/2)}$ of Eq. (9.32) by setting $F = 0$, and similarly for $I_{grav}^{(1/2)}$.

So eleven-dimensional supergravity on $M_{10} \times S^1/\mathbb{Z}_2$ is anomalous and has to be modified in order to be a consistent theory. An idea that has been very fruitful in string theory over the last years is to introduce new fields which live on the singularities of the space under consideration. Following this general tack we introduce massless modes living only on the fixed planes of our orbifold. These so-called twisted fields have to be ten-dimensional vector multiplets because the vector multiplet is the only ten-dimensional supermultiplet with all spins ≤ 1. In particular, the multiplets can be chosen in such a way that the gaugino fields have positive chirality (in the Minkowskian). Then they contribute to the pure and mixed gauge anomalies, as well as to the gravitational ones. The corresponding anomaly polynomial reads

$$I_{12(i)}^{(twisted)} = -\frac{1}{2}\left(n_i I_{grav}^{(1/2)}(R_i) + I_{mixed}^{(1/2)}(R_i, F_i) + I_{gauge}^{(1/2)}(F_i)\right) , \tag{12.5}$$

where the minus sign comes from the fact that the gaugino fields have negative chirality in the Euclidean. n_i is the dimension of the adjoint representation of the gauge group G_i. Adding all the pieces gives

$$\begin{aligned} I_{12(i)}^{(fields)} &= I_{12(i)}^{(untwisted)} + I_{12(i)}^{(twisted)} \\ &= -\frac{1}{2(2\pi)^5 6!}\left[\frac{496 - 2n_i}{1008}\mathrm{tr}R_i^6 + \frac{-224 - 2n_i}{768}\mathrm{tr}R_i^4\mathrm{tr}R_i^2 + \frac{320 - 10n_i}{9216}(\mathrm{tr}R_i^2)^3\right. \\ &\quad \left. + \frac{1}{16}\mathrm{tr}R_i^4\mathrm{Tr}F_i^2 + \frac{5}{64}(\mathrm{tr}R_i^2)^2\mathrm{Tr}F_i^2 - \frac{5}{8}\mathrm{tr}R_i^2\mathrm{Tr}F_i^4 + \mathrm{Tr}F_i^6\right] , \end{aligned} \tag{12.6}$$

where now Tr denotes the adjoint trace. To derive this formula we made use of the general form of the anomaly polynomial as given in appendix E. The anomaly cancels only if several conditions are met. First of all it is not possible to cancel the $\mathrm{tr}R^6$ term

by a Green-Schwarz type mechanism. Therefore, we get a restriction on the gauge group G_i, namely

$$n_i = 248. \tag{12.7}$$

Then we are left with

$$\begin{aligned} I_{12(i)}^{(fields)} &= -\frac{1}{2(2\pi)^5 6!}\left[-\frac{15}{16}\mathrm{tr}R_i^4\mathrm{tr}R_i^2 - \frac{15}{64}(\mathrm{tr}R_i^2)^3 + \frac{1}{16}\mathrm{tr}R_i^4\mathrm{Tr}F_i^2\right. \\ &\quad \left. + \frac{5}{64}(\mathrm{tr}R_i^2)^2\mathrm{Tr}F_i^2 - \frac{5}{8}\mathrm{tr}R_i^2\mathrm{Tr}F_i^4 + \mathrm{Tr}F_i^6\right] . \end{aligned} \tag{12.8}$$

In order to cancel this remaining part of the anomaly we will apply a sort of Green-Schwarz mechanism. This is possible if and only if the anomaly polynomial factorizes into the product of a four-form and an eight-form. For this factorization to occur we need

$$\mathrm{Tr}F_i^6 = \frac{1}{24}\mathrm{Tr}F_i^4\mathrm{Tr}F_i^2 - \frac{1}{3600}(\mathrm{Tr}F_i^2)^3 . \tag{12.9}$$

There is exactly one non-Abelian Lie group with this property, which is the exceptional group E_8. Defining $\mathrm{tr} := \frac{1}{30}\mathrm{Tr}$ for E_8 and making use of the identities

$$\mathrm{Tr}F^2 =: 30\ \mathrm{tr}F^2 , \tag{12.10}$$

$$\mathrm{Tr}F^4 = \frac{1}{100}(\mathrm{Tr}F^2)^2 , \tag{12.11}$$

$$\mathrm{Tr}F^6 = \frac{1}{7200}(\mathrm{Tr}F^2)^3 , \tag{12.12}$$

which can be shown to hold for E_8, we can see that the anomaly factorizes,

$$I_{12(i)}^{(fields)} = -\frac{\pi}{3}(I_{4(i)})^3 - I_{4(i)} \wedge X_8 , \tag{12.13}$$

with

$$I_{4(i)} := \frac{1}{16\pi^2}\left(\mathrm{tr}F_i^2 - \frac{1}{2}\mathrm{tr}R_i^2\right) , \tag{12.14}$$

and X_8 as in (10.15). X_8 is related to forms X_7 and X_6 via the usual descent mechanism $X_8 = dX_7,\ \delta X_7 = dX_6$.

The modified Bianchi identity

So far we saw that M-theory on $S^1/\mathbb{Z}_2$ is anomalous and we added new fields onto the fixed planes to cancel part of that anomaly. But now the theory has changed. It no longer is pure eleven-dimensional supergravity on a manifold with boundary, but we have to couple this theory to ten-dimensional super-Yang-Mills theory, with action

$$S_{SYM} = -\frac{1}{4\lambda^2}\int d^{10}x\sqrt{g_{10}}\ \mathrm{tr}\,F_{\mu\nu}F^{\mu\nu}, \tag{12.15}$$

where λ is an unknown coupling constant. The explicit coupling of these two theories was determined in [83]. The crucial result of this calculation is that the Bianchi identity, $\mathrm{d}G = 0$, needs to be modified. It reads[2]

$$\mathrm{d}G = -\frac{2\kappa_{11}^2}{\lambda^2} \sum_i \delta_i \wedge \left(\mathrm{tr}F_i^2 - \frac{1}{2}\mathrm{tr}R^2 \right) = -(4\pi)^2 \frac{2\kappa_{11}^2}{\lambda^2} \sum_i \delta_i \wedge I_{4(i)} \ . \tag{12.16}$$

Since δ_i has support only on the fixed planes and is a one-form $\sim \mathrm{d}\phi$, only the values of the smooth four-form $I_{4(i)}$ on this fixed plane are relevant and only the components not including $\mathrm{d}\phi$ do not vanish. The gauge part $\mathrm{tr}F_i^2$ always satisfies these conditions but for the $\mathrm{tr}R^2$ term this is non-trivial. In the following a bar on a form will indicate that all components containing $\mathrm{d}\phi$ are dropped and the argument is set to $\phi = \phi_i$. Then the modified Bianchi identity reads

$$\mathrm{d}G = \gamma \sum_i \delta_i \wedge \bar{I}_{4(i)} \ , \tag{12.17}$$

where we introduced

$$\gamma := -(4\pi)^2 \frac{2\kappa_{11}^2}{\lambda^2} \ . \tag{12.18}$$

Define the Chern-Simons form

$$\bar{\omega}_i := \frac{1}{(4\pi)^2} \left(\mathrm{tr}(A_i \mathrm{d}A_i + \frac{2}{3}A_i^3) - \frac{1}{2}\mathrm{tr}(\bar{\Omega}_i \mathrm{d}\bar{\Omega}_i + \frac{2}{3}\bar{\Omega}_i^3) \right) \ , \tag{12.19}$$

so that

$$\mathrm{d}\bar{\omega}_i = \bar{I}_{4(i)} \ . \tag{12.20}$$

Under a gauge and local Lorentz transformation with parameters Λ^g and Λ^L independent of ϕ one has

$$\delta\bar{\omega}_i = \mathrm{d}\bar{\omega}_i^1 \ , \tag{12.21}$$

where

$$\bar{\omega}_i^1 := \frac{1}{(4\pi)^2} \left(\mathrm{tr}\Lambda^g \mathrm{d}A_i - \frac{1}{2}\mathrm{tr}\Lambda^L \mathrm{d}\bar{\Omega}_i \right) . \tag{12.22}$$

Making use of (12.2) we find that the Bianchi identity (12.17) is solved by

$$G = \mathrm{d}C - (1-b)\gamma \sum_i \delta_i \wedge \bar{\omega}_i + b\gamma \sum_i \frac{\epsilon_i}{2} \bar{I}_{4(i)} - b\gamma \sum_i \frac{\mathrm{d}\phi}{2\pi} \wedge \bar{\omega}_i \ , \tag{12.23}$$

where b is an undetermined (real) parameter. As G is a physical field it is taken to be gauge invariant, $\delta G = 0$. Hence we get the transformation law of the C-field,

$$\delta C = \mathrm{d}B_2^1 - \gamma \sum_i \delta_i \wedge \bar{\omega}_i^1 - b\gamma \sum_i \frac{\epsilon_i}{2} \mathrm{d}\bar{\omega}_i^1 \ , \tag{12.24}$$

[2] This differs by a factor 2 from [83] which comes from the fact that our κ_{11} is the "downstairs" κ. [83] use its "upstairs" version and the relation between the two is $2\kappa_{downstairs} \equiv 2\kappa_{11} = \kappa_{upstairs}$. See [22] and [P3] for a careful discussion.

with some two-form B_2^1. Recalling that $C_{\mu\nu\rho}$ is projected out, this equation can be solved, because $C_{\mu\nu\rho} = 0$ is only reasonable if we also have $\delta C_{\mu\nu\rho} = 0$. This gives

$$(\mathrm{d}B_2^1)_{\mu\nu\rho} = \frac{b\gamma}{2} \sum_i (\epsilon_i \mathrm{d}\bar{\omega}_i^1)_{\mu\nu\rho} \ , \tag{12.25}$$

which is solved by

$$(B_2^1)_{\mu\nu} = \gamma \frac{b}{2} \sum_i \epsilon_i (\bar{\omega}_i^1)_{\mu\nu} \ . \tag{12.26}$$

So we choose

$$B_2^1 = \gamma \frac{b}{2} \sum_i \epsilon_i \bar{\omega}_i^1 \ , \tag{12.27}$$

and get

$$\delta C = \gamma \sum_i \left[(b-1)\delta_i \wedge \bar{\omega}_i^1 - \frac{b}{2\pi} \mathrm{d}\phi \wedge \bar{\omega}_i^1 \right] \ . \tag{12.28}$$

Inflow terms and anomaly cancellation

In the last sections we saw that introducing a vector supermultiplet cancels part of the gravitational anomaly that is present on the ten-dimensional fixed planes. Furthermore, the modified Bianchi identity led to a very special transformation law for the C-field. In this section we show that this modified transformation law allows us to cancel the remaining anomaly, leading to an anomaly free theory. We start from supergravity on $M_{10} \times I \cong M_{10} \times S^1/\mathbb{Z}_2$ and rewrite it in the upstairs formalism,

$$S_{top} = -\frac{1}{12\kappa_{11}^2} \int_{M_{10}\times I} C \wedge \mathrm{d}C \wedge \mathrm{d}C = -\frac{1}{24\kappa_{11}^2} \int_{M_{10}\times S^1} C \wedge \mathrm{d}C \wedge \mathrm{d}C \ . \tag{12.29}$$

However, we no longer have $G = \mathrm{d}C$ and thus it is no longer clear whether the correct topological term is $C\mathrm{d}C\mathrm{d}C$ or rather CGG. It turns out that the correct term is the one which maintains the structure $\widetilde{C}\mathrm{d}\widetilde{C}\mathrm{d}\widetilde{C}$ everywhere except on the fixed planes. However, the field C has to be modified to a field $\widetilde{C}$, similarly to what we did in chapter 10. To be concrete let us study the structure of G in more detail. It is given by

$$G = \mathrm{d}\left(C + \frac{b}{2}\gamma \sum_i \epsilon_i \bar{\omega}_i \right) - \gamma \sum_i \delta_i \wedge \bar{\omega}_i =: \mathrm{d}\widetilde{C} - \gamma \sum_i \delta_i \wedge \bar{\omega}_i \ . \tag{12.30}$$

That is we have $G = \mathrm{d}\widetilde{C}$ except on the fixed planes where we get an additional contribution. Thus, in order to maintain the structure of the topological term almost everywhere we postulate [P3] it to read

$$\begin{aligned} \widetilde{S}_{top} &= -\frac{1}{24\kappa_{11}^2} \int_{M_{10}\times S^1} \widetilde{C} \wedge G \wedge G \\ &= -\frac{1}{24\kappa_{11}^2} \int_{M_{10}\times S^1} \left(\widetilde{C} \wedge \mathrm{d}\widetilde{C} \wedge \mathrm{d}\widetilde{C} - 2\widetilde{C} \wedge \mathrm{d}\widetilde{C} \wedge \gamma \sum_i \delta_i \wedge \bar{\omega}_i \right) \ . \end{aligned} \tag{12.31}$$

To see that this is reasonable let us calculate its variation under gauge transformations. From (12.28) we have

$$\delta\widetilde{C} = \mathrm{d}\left(\frac{\gamma b}{2}\sum_i \epsilon_i \bar{\omega}_i^1\right) - \gamma \sum_i \delta_i \wedge \bar{\omega}_i^1 \ , \tag{12.32}$$

and we find

$$\delta\widetilde{S}_{top} = \frac{\gamma^3 b^2}{96\kappa_{11}^2}\int_{M_{10}\times S^1}\sum_{i,j,k}(\delta_i\epsilon_j\epsilon_k + 2\epsilon_i\epsilon_j\delta_k)\bar{\omega}_i^1\wedge\bar{I}_{4(j)}\wedge\bar{I}_{4(k)} = \frac{\gamma^3 b^2}{96\kappa_{11}^2}\sum_i\int \bar{\omega}_i^1\wedge\bar{I}_{4(i)}\wedge\bar{I}_{4(i)} \ , \tag{12.33}$$

where we used (12.3). This $\delta\tilde{S}_{top}$ is a sum of two terms, and each of them is localised on one of the fixed planes. The corresponding (Minkowskian) anomaly polynomial reads

$$I_{12}^{(top)} = \sum_i \frac{\gamma^3 b^2}{96\kappa_{11}^2}(\bar{I}_{4(i)})^3 =: \sum_i I_{12(i)}^{(top)} \ . \tag{12.34}$$

If we choose γ to be

$$\gamma = -\left(\frac{32\pi\kappa_{11}^2}{b^2}\right)^{1/3} \tag{12.35}$$

and use (9.36) we see that this cancels the first part of the anomaly (12.13) through inflow. Note that this amounts to specifying a certain choice for the coupling constant λ.

This does not yet cancel the anomaly entirely. However, we have seen, that there is yet another term, which can be considered as a first M-theory correction to eleven-dimensional supergravity, namely the Green-Schwarz term[3]

$$S_{GS} := -\frac{1}{(4\pi\kappa_{11}^2)^{1/3}}\int_{M_{10}\times I} G\wedge X_7 = -\frac{1}{2(4\pi\kappa_{11}^2)^{1/3}}\int_{M_{10}\times S^1} G\wedge X_7 \ , \tag{12.36}$$

studied in [131], [48], [P3]. X_8 is given in (10.15) and it satisfies the descent equations $X_8 = \mathrm{d}X_7$ and $\delta X_7 = \mathrm{d}X_6$. Its variation gives the final contribution to our anomaly,

$$\delta S_{GS} = -\frac{1}{2(4\pi\kappa_{11}^2)^{1/3}}\int_{M_{10}\times S^1} G\wedge \mathrm{d}X_6^1 = \sum_i \frac{\gamma}{2(4\pi\kappa_{11}^2)^{1/3}}\int_{M_{10}} \bar{I}_{4(i)}\wedge\bar{X}_6 \ , \tag{12.37}$$

where we integrated by parts and used the properties of the δ_i. The corresponding (Minkowskian) anomaly polynomial is

$$I_{12}^{(GS)} = \sum_i \frac{\gamma}{2(4\pi\kappa_{11}^2)^{1/3}}\ \bar{I}_{4(i)}\wedge\bar{X}_8 =: \sum_i I_{12(i)}^{(GS)} \ . \tag{12.38}$$

[3]The reader might wonder why we use the form $\int G\wedge X_7$ for the Green-Schwarz term, since we used $\int\tilde{C}\wedge X_8$ in chapter 10, c.f. Eq. (10.19). However, first of all we noted already in chapter 10 that $\int\tilde{G}\wedge X_7$ would have led to the same results. Furthermore, on $M_{10}\times S^1$ we have from (12.30) that $\int G\wedge X_7 = \int\tilde{C}\wedge X_8 - \gamma\sum_i\int\bar{\omega}_i\wedge X_7$. But the latter term is a local counterterm that does not contribute to I_{12}.

which cancels the second part of our anomaly provided

$$\gamma = -(32\pi\kappa_{11}^2)^{1/3} \ . \tag{12.39}$$

Happily, the sign is consistent with our first condition (12.35) for anomaly cancellation and it selects $b = 1$. This value for b was suggested in [22] from general considerations unrelated to anomaly cancellation. Choosing γ (and thus the corresponding value for λ) as in (12.39) leads to a local cancellation of the anomalies. Indeed let us collect all the contributions to the anomaly of a single fixed plane, namely (12.13), (12.34) and (12.38)

$$iI_{12(i)}^{(untwisted)} + iI_{12(i)}^{(twisted)} - iI_{12(i)}^{(top)} - iI_{12(i)}^{(GS)} = 0 \ . \tag{12.40}$$

The prefactors of $-i$ in the last two terms come from the fact that we calculated the variation of the Minkowskian action, which has to be translated to Euclidean space (c.f. Eq. 9.36). Note that the anomalies cancel separately on each of the two ten-dimensional planes. In other words, we once again find *local* anomaly cancellation.

Chapter 13

Conclusions

We have seen that anomalies are a powerful tool to explore some of the phenomena of M-theory. The requirement of a cancellation of local gauge and gravitational anomalies imposes strong constraints on the theory, and allows us to understand its structure in more detail. In the context of higher dimensional field theories anomalies can cancel in two different ways. For instance, in the case of M-theory on the product of Minkowski space with a compact manifold, anomalies can be localised at various points in the internal space. The requirement of global anomaly cancellation then simply means that the sum of all these anomalies has to vanish. The much stronger concept of local anomaly cancellation, on the other hand, requires the anomaly to be cancelled on the very space where it is generated. We have seen that for M-theory on singular G_2-manifolds and on $M_{10} \times S^1/\mathbb{Z}_2$ anomalies do indeed cancel locally via a mechanism known as anomaly inflow. The main idea is that the classical action is not invariant under local gauge or Lorentz transformations, because of "defects" in the space on which it is formulated. Such a defect might be a (conical or orbifold) singularity or a boundary. Furthermore, we saw that the classical action had to be modified close to these defects. Only then does the variation of the action give the correct contributions to cancel the anomaly. These modifications of the action in the cases studied above are modelled after the similar methods that had been used in [60] to cancel the normal bundle anomaly of the M5-brane. Of course these modifications are rather ad hoc. Although the same method seems to work in many different cases the underlying physics has not yet been understood. One might for example ask how the smooth function ρ (c.f Eq. (10.3)) is generated in the context of G_2-compactifications. Some progress in this direction has been made in [77], [24]. It would certainly be quite interesting to further explore these issues.

There are no explicit examples of metrics of compact G_2-manifolds with conical singularities and therefore the discussion above might seem quite academic. However, we were able to write down relatively simple metrics for a compact manifold with two conical singularities and weak G_2-holonomy. Although the corresponding effective theory lives on AdS_4 one expects that the entire mechanism of anomaly cancellation should be applicable to this case as well, see e.g. [5] for a discussion. Therefore, our explicit weak G_2-metric might serve as a useful toy model for the full G_2-case.

Part III

Appendices

Appendix A

Notation

Our notation is as in [P4]. However, for the reader's convenience we list the relevant details once again.

A.1 General notation

The metric on flat space is given by

$$\eta := \mathrm{diag}(-1, 1, \dots 1) \ . \tag{A.1}$$

The anti-symmetric tensor is defined as

$$\widetilde{\epsilon}_{012\dots d-1} \ := \ \widetilde{\epsilon}^{\,012\dots d-1} := +1 \ , \tag{A.2}$$

$$\epsilon_{M_1\dots M_d} \ := \ \sqrt{g}\, \widetilde{\epsilon}_{M_1\dots M_d} \ . \tag{A.3}$$

That is, we define $\widetilde{\epsilon}$ to be the tensor density and ϵ to be the tensor. We obtain

$$\epsilon_{012\dots d-1} \ = \ \sqrt{g} = e := |\det e^A{}_M| \ , \tag{A.4}$$

$$\epsilon^{M_1\dots M_d} \ = \ \mathrm{sig}(g)\frac{1}{\sqrt{g}}\, \widetilde{\epsilon}^{\,M_1\dots M_d} \ , \quad \text{and} \tag{A.5}$$

$$\widetilde{\epsilon}^{\,M_1\dots M_r P_1\dots P_{d-r}} \widetilde{\epsilon}_{N_1\dots N_r P_1\dots P_{d-r}} \ = \ r!(d-r)!\delta^{[M_1\dots M_r]}_{N_1\dots N_r} \ . \tag{A.6}$$

(Anti-)Symmetrisation is defined as,

$$A_{(M_1\dots M_l)} \ := \ \frac{1}{l!}\sum_\pi A_{M_{\pi(1)}\dots M_{\pi(l)}} \ , \tag{A.7}$$

$$A_{[M_1\dots M_l]} \ := \ \frac{1}{l!}\sum_\pi \mathrm{sig}(\pi) A_{M_{\pi(1)}\dots M_{\pi(l)}} \ . \tag{A.8}$$

p-forms come with a factor of $p!$, e.g.

$$\omega := \frac{1}{p!}\omega_{M_1\dots M_p} dz^{M_1} \wedge \dots \wedge dz^{M_p} \ . \tag{A.9}$$

The Hodge dual is defined as

$$*\omega = \frac{1}{p!(d-p)!}\,\omega_{M_1\dots M_p}\,\epsilon^{M_1\dots M_p}{}_{M_{p+1}\dots M_d}\,dz^{M_{p+1}}\wedge\dots\wedge dz^{M_d}\ . \tag{A.10}$$

A.2 Spinors

A.2.1 Clifford algebras and their representation

Definition A.1 A *Clifford algebra* in d dimensions is defined as a set containing d elements Γ^A which satisfy the relation

$$\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}\mathbb{1}\ . \tag{A.11}$$

Under multiplication this set generates a finite group, denoted C_d, with elements

$$C_d = \{\pm 1, \pm\Gamma^A, \pm\Gamma^{A_1A_2}, \dots, \pm\Gamma^{A_1\dots A_d}\}\ , \tag{A.12}$$

where $\Gamma^{A_1\dots A_l} := \Gamma^{[A_1}\dots\Gamma^{A_l]}$. The order of this group is

$$\mathrm{ord}(C_d) = 2\sum_{p=0}^{d}\binom{d}{p} = 2\cdot 2^d = 2^{d+1}\ . \tag{A.13}$$

Definition A.2 Let G be a group. Then the *conjugacy class* $[a]$ of $a \in G$ is defined as

$$[a] := \{gag^{-1}|g \in G\}. \tag{A.14}$$

Proposition A.3 *Let G be a finite dimensional group. Then the number of its irreducible representations equals the number of its conjugacy classes.*

Definition A.4 Let G be a finite group. Then the *commutator group* $\mathrm{Com}(G)$ of G is defined as

$$\mathrm{Com}(G) := \{ghg^{-1}h^{-1}|g,h \in G\}\ . \tag{A.15}$$

Proposition A.5 *Let G be a finite group. Then the number of inequivalent one-dimensional representations is equal to the order of G divided by the order of the commutator group of G.*

Proposition A.6 *Let G be a finite group with inequivalent irreducible representations of dimension n_p, where p labels the representation. Then we have*

$$\mathrm{ord}(G) = \sum_p (n_p)^2\ . \tag{A.16}$$

Proposition A.7 *Every class of equivalent representations of a finite group G contains a unitary representation.*

For the unitary choice we get $\Gamma^A\Gamma^{A\dagger} = \mathbb{1}$. From (A.11) we infer (in Minkowski space) $\Gamma^{0\dagger} = -\Gamma^0$ and $(\Gamma^A)^\dagger = \Gamma^A$ for $A \neq 0$. This can be rewritten as

$$\Gamma^{A\dagger} = \Gamma^0\Gamma^A\Gamma^0\ . \tag{A.17}$$

Clifford algebras in even dimensions

Theorem A.8 *For $d = 2k+2$ even the group C_d has $2^d + 1$ inequivalent representations. Of these 2^d are one-dimensional and the remaining representation has (complex) dimension $2^{\frac{d}{2}} = 2^{k+1}$.*

This can be proved by noting that for even d the conjugacy classes of C_d are given by

$$\left\{[+1], [-1], [\Gamma^A], [\Gamma^{A_1 A_2}], \ldots, [\Gamma^{A_1 \ldots A_d}]\right\} ,$$

hence the number of inequivalent irreducible representations of C_d is $2^d + 1$. The commutator of C_d is $\mathrm{Com}(C_d) = \{\pm 1\}$ and we conclude that the number of inequivalent one-dimensional representations of C_d is 2^d. From (A.16) we read off that the dimension of the remaining representation has to be $2^{\frac{d}{2}}$.

Having found irreducible representations of C_d we turn to the question whether we also found representations of the Clifford algebra. In fact, for elements of the Clifford algebra we do not only need the group multiplication, but the addition of two elements must be well-defined as well, in order to make sense of (A.11). It turns out that the one-dimensional representation of C_d do not extend to representations of the Clifford algebra, as they do not obey the rules for addition and subtraction. Hence, we found that for d even there is a unique class of irreducible representations of the Clifford algebra of dimension $2^{\frac{d}{2}} = 2^{k+1}$.

Given an irreducible representation $\{\Gamma^A\}$ of a Clifford algebra, it is clear that $\pm\{\Gamma^{A*}\}$ and $\pm\{\Gamma^{A\tau}\}$ form irreducible representations as well. As there is a unique class of representations in even dimensions, these have to be related by similarity transformations,

$$\begin{aligned} \Gamma^{A*} &= \pm(B_\pm)^{-1}\Gamma^A B_\pm \; , \\ \Gamma^{A\tau} &= \pm(C_\pm)^{-1}\Gamma^A C_\pm \; . \end{aligned} \tag{A.18}$$

The matrices $C_\pm$ are known as *charge conjugation matrices*. Iterating this definition gives conditions for $B_\pm, C_\pm$,

$$(B_\pm)^{-1} = b_\pm B_\pm{}^* \; , \tag{A.19}$$

$$C_\pm = c_\pm C_\pm{}^\tau \; , \tag{A.20}$$

with $b_\pm$ real, $c_\pm \in \{\pm 1\}$ and $C_\pm$ symmetric or anti-symmetric.

Clifford algebras in odd dimensions

Theorem A.9 *For $d = 2k+3$ odd the group C_d has 2^d+2 inequivalent representations. Of these 2^d are one-dimensional*[1] *and the remaining two representation have (complex) dimension $2^{\frac{d-1}{2}} = 2^{k+1}$.*

[1] As above these will not be considered any longer as they are representations of C_d but not of the Clifford algebra.

As above we note that for odd d the conjugacy classes of C_d are given by

$$\{[+1], [-1], [\Gamma^A], [\Gamma^{A_1A_2}], \ldots, [\Gamma^{A_1\ldots A_d}], [-\Gamma^{A_1\ldots A_d}]\}$$

and the number of inequivalent irreducible representations of C_d is 2^d+2. Again we find the commutator $\mathrm{Com}(C_d) = \{\pm 1\}$, hence, the number of inequivalent one-dimensional representations of C_d is 2^d. Now define the matrix

$$\Gamma^d := \Gamma^0\Gamma^1 \ldots \Gamma^{d-1} \ , \tag{A.21}$$

which commutes with all elements of C_d. By Schur's lemma this must be a multiple of the identity, $\Gamma^d = a^{-1}\mathbb{1}$, with some constant a. Multiplying by Γ^{d-1} we find

$$\Gamma^{d-1} = a\Gamma^0\Gamma^1 \ldots \Gamma^{d-2} \ . \tag{A.22}$$

Furthermore, $(\Gamma^0\Gamma^1 \ldots \Gamma^{d-2})^2 = -(-1)^{k+1}$. As we know from (A.11) that $(\Gamma^{d-1})^2 = +1$ we conclude that $a = \pm 1$ for $d = 3 \pmod 4$ and $a = \pm i$ for $d = 5 \pmod 4$. The matrices $\{\Gamma^0, \Gamma^1, \ldots, \Gamma^{d-2}\}$ generate an even-dimensional Clifford algebra the dimension of which has been determined to be 2^{k+1}. Therefore, the two inequivalent irreducible representations of C_d for odd d must coincide with this irreducible representation when restricted to C_{d-1}. We conclude that the two irreducible representations for C_d and odd d are generated by the unique irreducible representation for $\{\Gamma^0, \Gamma^1, \Gamma^{d-2}\}$, together with the matrix $\Gamma^{d-1} = a\Gamma^0\Gamma^1 \ldots \Gamma^{d-2}$. The two possible choices of a correspond to the two inequivalent representations. The dimension of these representation is 2^{k+1}.

A.2.2 Dirac, Weyl and Majorana spinors

Dirac spinors

Let (M, g) be an oriented pseudo-Riemannian manifold of dimension d, which is identified with d-dimensional space-time, and let $\{\Gamma^A\}$ be a d-dimensional Clifford algebra. The metric and orientation induce a unique $SO(d-1,1)$-structure[2] P on M. A *spin structure* $(\widetilde{P}, \pi)$ on M is a principal bundle $\widetilde{P}$ over M with fibre $\mathrm{Spin}(d-1,1)$, together with a map of bundles $\pi : \widetilde{P} \to P$. $\mathrm{Spin}(d-1,1)$ is the universal covering group of $SO(d-1,1)$. Spin structures do not exist on every manifold. An oriented pseudo-Riemannian manifold M admits a spin structure if and only if $w_2(M) = 0$, where $w_2(M) \in H^2(M, \mathbb{Z})$ is the *second Stiefel-Whitney class* of M. In that case we call M a *spin manifold*.

Define the anti-Hermitian generators

$$\Sigma^{AB} := \frac{1}{2}\Gamma^{AB} = \frac{1}{4}[\Gamma^A, \Gamma^B] \ . \tag{A.23}$$

[2]Let M be a manifold of dimension d, and F the frame bundle over M. Then F is a principle bundle over M with structure group $GL(d, \mathbb{R})$. A *G-structure* on M is a principle subbundle P of F with fibre G.

Then the Σ^{AB} form a representation of $\mathtt{so}(d-1,1)$, the Lie algebra of $SO(d-1,1)$,

$$[\Sigma^{AB},\Sigma^{CD}] = -\Sigma^{AC}g^{BD} + \Sigma^{AD}g^{BC} + \Sigma^{BC}g^{AD} - \Sigma^{BD}g^{AC} \ . \tag{A.24}$$

In fact, Σ^{AB} are generators of $\mathrm{Spin}(d-1,1)$. Take Δ^d to be the natural representation of $\mathrm{Spin}(d-1,1)$. We define the *(complex) spin bundle* $S \to M$ to be[3] $S := \widetilde{P}\times_{\mathrm{Spin}(d-1,1)}\Delta^d$. Then S is a complex vector bundle over M, with fibre Δ^d of dimension $2^{[d/2]}$. A *Dirac spinor* ψ is defined as a section of the spin bundle S. Under a local Lorentz transformation with infinitesimal parameter $\alpha_{AB} = -\alpha_{BA}$ a Dirac spinor transforms as

$$\psi' = \psi + \delta\psi = \psi - \frac{1}{2}\alpha_{AB}\Sigma^{AB}\psi \ . \tag{A.25}$$

The *Dirac conjugate* $\bar{\psi}$ of the spinor ψ is defined as

$$\bar{\psi} := i\psi^\dagger\Gamma^0 \ . \tag{A.26}$$

With this definition we have $\delta(\bar{\psi}\eta) = 0$ and $\bar{\psi}\psi$ is Hermitian, $(\bar{\psi}\psi)^\dagger = \bar{\psi}\psi$.

Weyl spinors

In $d = 2k+2$-dimensional space-time we can construct the matrix[4]

$$\Gamma_{d+1} := (-i)^k\Gamma^0\Gamma^1\ldots\Gamma^{d-1} \ , \tag{A.27}$$

which satisfies

$$(\Gamma_{d+1})^2 = \mathbb{1} \ , \tag{A.28}$$

$$\{\Gamma_{d+1},\Gamma^A\} = 0 \ , \tag{A.29}$$

$$[\Gamma_{d+1},\Sigma^{AB}] = 0 \ . \tag{A.30}$$

Then, we can define the chirality projectors

$$P_L \equiv P_- := \frac{1}{2}(\mathbb{1} - \Gamma_{d+1}) \quad , \quad P_R \equiv P_+ := \frac{1}{2}(\mathbb{1} + \Gamma_{d+1}) \ , \tag{A.31}$$

satisfying

$$\begin{aligned} P_L + P_R &= \mathbb{1} \ , \\ P_{L,R}^2 &= P_{L,R} \ , \\ P_LP_R &= P_RP_L = 0 \ , \\ [P_{L,R},\Sigma^{AB}] &= 0 \ . \end{aligned} \tag{A.32}$$

[3] $\widetilde{P}\times_{\mathrm{Spin}(d-1,1)}\Delta^d$ is the fibre bundle which is associated to the principal bundle $\widetilde{P}$ in a natural way. Details of this construction can be found in any textbook on differential geometry, see for example [109].

[4] This definition of the Γ-matrix in Minkowski space agrees with the one of [117]. Sometimes it is useful to define a Minkowskian Γ-matrix as $\Gamma = i^k\Gamma^0\ldots\Gamma^{d-1}$ as in [P3]. Obviously the two conventions agree in 2, 6 and 10 dimensions and differ by a sign in dimensions 4 and 8.

A *Weyl spinor* in even-dimensional spaces is defined as a spinor satisfying the Weyl condition,

$$P_{L,R}\psi = \psi \ . \tag{A.33}$$

Note that this condition is Lorentz invariant, as the projection operators commute with Σ^{AB}. Spinors satisfying $P_L\psi_L = \psi_L$ are called *left-handed* Weyl spinors and those satisfying $P_R\psi_R = \psi_R$ are called *right-handed*. The Weyl condition reduces the number of complex components of a spinor to 2^k.

Obviously, under the projections $P_{L,R}$ the space Δ^d splits into a direct sum $\Delta^d = \Delta^d_+ \oplus \Delta^d_-$ and the spin bundle is given by the Whitney sum $S = S_+ \oplus S_-$. Left- and right-handed Weyl spinors are sections of S_- and S_+, respectively.

Majorana spinors

In (A.18) we defined the matrices $B_\pm$ and $C_\pm$. We now want to explore these matrices in more detail. For $d = 2k+2$ we define *Majorana spinors* as those spinors that satisfy

$$\psi = B_+\psi^* \ , \tag{A.34}$$

and pseudo-Majorana spinors as those satisfying

$$\psi = B_-\psi^* \ . \tag{A.35}$$

As in the case of the Weyl conditions, these conditions reduce the number of components of a spinor by one half. The definitions imply

$$\begin{aligned} B_+^*B_+ &= \mathbb{1} \ , \\ B_-^*B_- &= \mathbb{1} \ , \end{aligned} \tag{A.36}$$

which in turn would give $b_+ = 1$ and $b_- = 1$. These are non-trivial conditions since $B_\pm$ is fixed by its definition (A.18). The existence of (pseudo-) Majorana spinors relies on the possibility to construct matrices B_+ or B_- which satisfy (A.36). It turns out that Majorana conditions can be imposed in 2 and 4 (mod 8) dimensions. Pseudo-Majorana conditions are possible in 2 and 8 (mod 8) dimensions.

Finally, we state that in odd dimensions Majorana spinors can be defined in dimensions 3 (mod 8) and pseudo-Majorana in dimension 1 (mod 8)

Majorana-Weyl spinors

For $d = 2k+2$ dimensions one might try to impose both the Majorana (or pseudo-Majorana) and the Weyl condition. This certainly leads to spinors with 2^{k-1} components. From (A.27) we get $(\Gamma_{d+1})^* = (-1)^k B_\pm^{-1}\Gamma_{d+1}B_\pm$ and therefore for $d = 2$ (mod 4)

$$P_{L,R}^* = B_\pm^{-1}P_{L,R}B_\pm \ , \tag{A.37}$$

and for $d = 4$ (mod 4)

$$P_{L,R}^* = B_\pm^{-1}P_{R,L}B_\pm \ . \tag{A.38}$$

But this implies that imposing both the Majorana and the Weyl condition is consistent only in dimensions $d = 2 \pmod 4$, as we get

$$B_\pm (P_{L,R}\psi)^* = P_{L,R} B_\pm \psi^* \ ,$$

for $d = 2 \pmod 4$, but

$$B_\pm (P_{L,R}\psi)^* = P_{R,L} B_\pm \psi^* \tag{A.39}$$

for $d = 4 \pmod 4$. We see that in the latter case the operator $B_\pm$ is a map between states of different chirality, which is inconsistent with the Weyl condition. As the Majorana condition can be imposed only in dimensions 2, 4 and 8 (mod 8) we conclude that (pseudo-) Majorana-Weyl spinors can only exist in dimensions 2 (mod 8).

We summarize the results on spinors in various dimensions in the following table.

d	Dirac	Weyl	Majorana	Pseudo-Majorana	Majorana-Weyl
2	4	2	2	2	1
3	4		2		
4	8	4	4		
5	8				
6	16	8			
7	16				
8	32	16		16	
9	32			16	
10	64	32	32	32	16
11	64		32		
12	128	64	64		

The numbers indicate the real dimension of a spinor, whenever it exists.

A.3 Gauge theory

Gauge theories are formulated on principal bundles $P \to M$ on a base space M with fibre G known as the *gauge group*. Any group element g of the connected component of G that contains the unit element can be written as $g := e^\Lambda$, with $\Lambda := \Lambda_a T_a$ and T_a basis vectors of the Lie algebra $\mathsf{g} := Lie(G)$. We always take T_a to be anti-Hermitian, s.t. $T_a =: -it_a$ with t_a Hermitian. The elements of a Lie algebra satisfy commutation relations

$$[T_a, T_b] = C^c{}_{ab} T_c \quad , \quad [t_a, t_b] = iC^c{}_{ab} t_c \ , \tag{A.40}$$

with the real valued *structure coefficients* $C^c{}_{ab}$.

Of course, the group G can come in various representations. The *adjoint representation*

$$(T^{Ad}_a)^b{}_c := -C^b{}_{ca} \tag{A.41}$$

is particularly important. Suppose a connection is given on the principal bundle. This induces a local Lie algebra valued connection form $A = A_a T_a$ and the corresponding local form of the curvature, $F = F_a T_a$. These forms are related by[5]

$$\begin{aligned} F &:= dA + \frac{1}{2}[A, A] \ , \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \ , \\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + C^a{}_{bc} A^b_\mu A^c_\nu \ . \end{aligned} \tag{A.42}$$

In going from one chart to another they transform as

$$A^g := g^{-1}(A + d)g \quad , \quad F^g := dA^g + \frac{1}{2}[A^g, A^g] = g^{-1} F g \ . \tag{A.43}$$

For $g = e^\epsilon = e^{\epsilon_a T_a}$ with ϵ infinitesimal we get $A^g = A + D\epsilon$.

For any object on the manifold which transforms under some representation $\widetilde{T}$ (with $\widetilde{T}$ anti-Hermitian) of the gauge group G we define a gauge covariant derivative

$$D := d + A \quad , \quad A := A_a \widetilde{T}_a \ . \tag{A.44}$$

When acting on Lie algebra valued fields the covariant derivative is understood to be $D := d + [A, \ ]$.

We have the general operator formula

$$DD\phi = F\phi \ , \tag{A.45}$$

which reads in components

$$[D_M, D_N]\phi = F_{MN}\phi \ . \tag{A.46}$$

Finally, we note that the curvature satisfies the Bianchi identity,

$$DF = 0 \ . \tag{A.47}$$

A.4 Curvature

Usually general relativity on a manifold M of dimension d is formulated in a way which makes the invariance under diffeomorphisms, $Diff(M)$, manifest. Its basic objects are tensors which transform covariantly under $GL(d, \mathbb{R})$. However, since $GL(d, \mathbb{R})$ does not admit a spinor representation, the theory has to be reformulated if we want to couple spinors to a gravitational field. This is done by choosing an orthonormal basis in the tangent space TM, which is different from the one induced by the coordinate system. From that procedure we get an additional local Lorentz invariance of the theory. As $SO(d-1, 1)$ does have a spinor representation we can couple spinors to this reformulated theory.

[5] The commutator of Lie algebra valued forms A and B is understood to be $[A, B] := [A_M, B_N]\ dz^M \wedge dz^N$.

At a point x on a pseudo-Riemannian manifold (M,g) we define the vielbeins $e_A(x)$ as

$$e_A(x) := e_A{}^M(x)\partial_M \ , \tag{A.48}$$

with coefficients $e_A{}^M(x)$ such that the $\{e_A\}$ are orthogonal,

$$g(e_A, e_B) = e_A{}^M e_B{}^N g_{MN} = \eta_{AB} \ . \tag{A.49}$$

Define the inverse coefficients via $e_A{}^M e^B{}_M = \delta_A^B$ and $e^A{}_M e_A{}^N = \delta_M^N$, which gives $g_{MN}(x) = \eta_{AB} e^A{}_M(x) e^B{}_N(x)$. The dual basis $\{e^A\}$ is defined as, $e^A := e^A{}_M dz^M$. The commutator of two vielbeins defines the *anholonomy coefficients* $\Omega_{AB}{}^C$,

$$[e_A, e_B] := [e_A{}^M \partial_M, e_B{}^N \partial_N] = \Omega_{AB}{}^C e_C \ , \tag{A.50}$$

and from the definition of e_A one can read off

$$\Omega_{AB}{}^C(x) = e^C{}_N[e_A{}^K(\partial_K e_B{}^N) - e_B{}^K(\partial_K e_A{}^N)](x) \ . \tag{A.51}$$

When acting on tensors expressed in the orthogonal basis, the covariant derivative has to be rewritten using the *spin connection coefficients* $\omega_M{}^A{}_B$,

$$\nabla^S_M V^{AB\ldots}_{CD\ldots} := \partial_M V^{AB\ldots}_{CD\ldots} + \omega_M{}^A{}_E V^{EB\ldots}_{CD\ldots} + \ldots - \omega_M{}^E{}_C V^{AB\ldots}_{ED\ldots} \ . \tag{A.52}$$

The object ∇^S is called the *spin connection*[6]. Its action can be extended to objects transforming under an arbitrary representation of the Lorentz group. Take a field ϕ which transforms as

$$\delta\phi^i = -\frac{1}{2}\epsilon_{AB}(T^{AB})^i{}_j\phi^j \tag{A.53}$$

under the infinitesimal Lorentz transformation $\Lambda^A{}_B(x) = \delta^A_B + \epsilon^A{}_B = \delta^A_B + \frac{1}{2}\epsilon_{CD}(T^{CD}_{vec})^A{}_B$, with the vector representation $(T^{CD}_{vec})^A{}_B = (\eta^{CA}\delta^D_B - \eta^{DA}\delta^C_B)$. Then its covariant derivative is defined as

$$\nabla^S_M\phi^i := \partial_M\phi^i + \frac{1}{2}\omega_{MAB}(T^{AB})^i{}_j\phi^j \ . \tag{A.54}$$

We see that the spin connection coefficients can be interpreted as the gauge field corresponding to local Lorentz invariance. Commuting two covariant derivatives gives the general formula

$$[\nabla^S_M, \nabla^S_N]\phi = \frac{1}{2}R_{MNAB}T^{AB}\phi \ . \tag{A.55}$$

In particular we can construct a connection on the spin bundle S of M. As we know that for $\psi \in C^\infty(S)$ the transformation law reads (with Σ^{AB} as defined in (A.24))

$$\delta\psi = -\frac{1}{2}\epsilon_{AB}\Sigma^{AB}\psi \ , \tag{A.56}$$

we find

$$\nabla^S_M\psi = \partial_M\psi + \frac{1}{2}\omega_{MAB}\Sigma^{AB}\psi = \partial_M\psi + \frac{1}{4}\omega_{MAB}\Gamma^{AB}\psi \ . \tag{A.57}$$

[6]Physicists usually use the term "spin connection" for the connection coefficients.

If we commute two spin connections acting on spin bundles we get

$$[\nabla^S_M, \nabla^S_N]\psi = \frac{1}{4} R_{MNAB}\Gamma^{AB}\psi, \tag{A.58}$$

where R is the curvature corresponding to ω, i.e. $R^A{}_B = D(\omega_M{}^A{}_B dz^M)$.

In the vielbein formalism the property $\nabla g_{MN} = 0$ translates to

$$\nabla^S_N e^A{}_M = 0 \ . \tag{A.59}$$

In the absence of torsion this gives the dependence of ω_{MAB} on the vielbeins. It can be expressed most conveniently using the anholonomy coefficients

$$\omega_{MAB}(e) = \frac{1}{2}(-\Omega_{MAB} + \Omega_{ABM} - \Omega_{BMA}) \ . \tag{A.60}$$

If torsion does not vanish one finds

$$\omega_{MAB} = \omega_{MAB}(e) + \kappa_{MAB} \ , \tag{A.61}$$

where κ_{MAB} is the contorsion tensor. It is related to the torsion tensor $\mathcal{T}$ by

$$\kappa_{MAB} = \mathcal{T}^L_{MN}(e_{AL}e_B{}^N - e_{BL}e_A{}^N) + g_{ML}\mathcal{T}^L_{NR}e_A{}^N e_B{}^R \ . \tag{A.62}$$

Defining $\omega^A{}_B := \omega_M{}^A{}_B dz^M$ one can derive the *Maurer-Cartan structure equations*,

$$de^A + \omega^A{}_B \wedge e^B = \mathcal{T}^A \quad , \quad d\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B = R^A{}_B \ , \tag{A.63}$$

where

$$\mathcal{T}^A = \frac{1}{2}\mathcal{T}^A{}_{MN} dz^M \wedge dz^N \quad , \quad R^A{}_B = \frac{1}{2} R^A{}_{BMN} dz^M \wedge dz^N \ , \tag{A.64}$$

and

$$\mathcal{T}^A{}_{MN} = e^A{}_P T^P{}_{MN} \quad , \quad R^A{}_{BMN} = e^A{}_Q e_B{}^P R^Q{}_{PMN} \ . \tag{A.65}$$

These equations tell us that the curvature corresponding to ∇ and the one corresponding to ∇^S are basically the same. The Maurer-Cartan structure equations can be rewritten as

$$\mathcal{T} = De \quad , \quad R = D\omega \ , \tag{A.66}$$

where $D = d + \omega$. $\mathcal{T}$ and R satisfy the Bianchi identities

$$D\mathcal{T} = Re \quad , \quad DR = 0 \ . \tag{A.67}$$

The Ricci tensor $\mathcal{R}_{MN}$ and the Ricci scalar $\mathcal{R}$ are given by

$$\mathcal{R}_{MN} := R_{MPNQ}g^{PQ} \ , \mathcal{R} := \mathcal{R}_{MN}g^{MN} \ . \tag{A.68}$$

Finally, we note that general relativity is a gauge theory in the sense of appendix A.3. If we take the induced basis as a basis for the tangent bundle the relevant group is $GL(d, \mathbb{R})$. If on the other hand we use the vielbein formalism the gauge group is $SO(d-1, 1)$. The gauge fields are Γ and ω, respectively. The curvature of these one-forms is the Riemann curvature tensor and the curvature two-form, respectively. However, general relativity is a very special gauge theory, as its connection coefficients can be constructed from another basic object on the manifold, namely the metric tensor g_{MN} or the vielbein $e_A{}^M$.

Appendix B

Some Mathematical Background

B.1 Useful facts from complex geometry

Let X be a complex manifold and define $\Lambda^{p,q}(X) := \Lambda^p((T^{(1,0)}X)^*) \otimes \Lambda^q((T^{(0,1)}X)^*)$. Then we have the decomposition

$$\Lambda^k(T^*X) \otimes \mathbb{C} = \bigoplus_{j=0}^{k} \Lambda^{j,k-j}(X) \ . \tag{B.1}$$

$\Lambda^{p,q}(X)$ are complex vector bundles, but in general they are not holomorphic vector bundles. One can show that the only holomorphic vector bundles are those with $q = 0$, i.e. $\Lambda^{0,0}(X), \Lambda^{1,0}(X), \ldots \Lambda^{m,0}(X)$, where $m := \dim_{\mathbb{C}} X$, [85]. Let s be a smooth section of $\Lambda^{p,0}(X)$ on X. s is a holomorphic section if and only if

$$\bar{\partial} s = 0 \ . \tag{B.2}$$

Such a holomorphic section is called a *holomorphic p-form*. Thus, the Dolbeault group $H^{(p,0)}_{\bar{\partial}}(X)$ is the vector space of holomorphic p-forms on X.

Definition B.1 Let X be a complex three manifold and $p \in X$. The *small resolution of X in p* is given by the pair $(\tilde{X}, \pi)$ defined s.t.

$$\begin{array}{rcll} \pi : \tilde{X} & \rightarrow & X & \\ \pi : \tilde{X} \backslash \pi^{-1}(p) & \rightarrow & X \backslash \{p\} & \text{is one to one.} \\ \pi^{-1}(p) & \cong & S^2 & \end{array} \tag{B.3}$$

The moduli space of complex structures

Proposition B.2 *The tangent space of the moduli space of complex structures $\mathcal{M}_{cs}$ of a complex manifold X is isomorphic to $H^1_{\bar{\partial}}(TX)$.*

Loosely speaking, the complex structure on a manifold X with coordinates $z^i, \bar{z}^{\bar{i}}$ tells us which functions are holomorphic, $\bar{\partial} f(z, \bar{z}) = 0$. Changing the complex structure therefore amounts to changing the operator $\bar{\partial} := \mathrm{d}\bar{z}^{\bar{i}} \partial_{\bar{z}^{\bar{i}}}$. Consider

$$\bar{\partial} \to \bar{\partial}' := \bar{\partial} + A \tag{B.4}$$

where $A := A^i \partial_i = A^i \partial_{z^i}$ and A^i is a one-form. Then linearizing $(\bar{\partial} + A)^2 = 0$ gives $\bar{\partial} A = 0$. On the other hand a change of coordinates $(z, \bar{z}) \to (w, \bar{w})$ with $z^i = w^i + v^i(\bar{w}^{\bar{j}})$, $\bar{z}^{\bar{i}} = \bar{w}^{\bar{i}}$ leads to

$$\bar{\partial} \to \bar{\partial}' = \bar{\partial} + (\bar{\partial} v^i) \partial_i \ . \tag{B.5}$$

This means that those transformations of the operator $\bar{\partial}$ that are exact, i.e. $A^i = \bar{\partial} v^i$, can be undone by a coordinate transformations. For A^i closed but not exact on the other hand, one changes the complex structure of the manifold. The corresponding A lie in $H^1_{\bar{\partial}}(TX)$ which was to be shown.

B.2 The theory of divisors

The concept of a divisor is a quite general and powerful tool in algebraic geometry. We will only be interested in divisors of forms and functions on compact Riemann surfaces. Indeed, the theory of divisors is quite convenient to keep track of the position and degree of zeros and poles on a Riemann surface. The general concept is defined in [71], we follow the exposition of [55].

Let then Σ be a compact Riemann surface of genus $\hat{g}$.

Definition B.3 A *divisor* on Σ is a formal symbol

$$\mathbf{A} = P_1^{\alpha_1} \dots P_k^{\alpha_k} \ , \tag{B.6}$$

with $P_j \in \Sigma$ and $\alpha_j \in \mathbb{Z}$.

This can be rewritten as

$$\mathbf{A} = \prod_{P \in \Sigma} P^{\alpha(P)} \ , \tag{B.7}$$

with $\alpha(P) \in \mathbb{Z}$ and $\alpha(P) \neq 0$ for only finitely many $P \in \Sigma$. The divisors on Σ form a group, $\mathrm{Div}(\Sigma)$, if we define the multiplication of $\mathbf{A}$ with

$$\mathbf{B} = \prod_{P \in \Sigma} P^{\beta(P)} \ , \tag{B.8}$$

by

$$\mathbf{AB} := \prod_{P \in \Sigma} P^{\alpha(P) + \beta(P)} \ , \tag{B.9}$$

The inverse of $\mathbf{A}$ is given by

$$\mathbf{A}^{-1} = \prod_{P \in \Sigma} P^{-\alpha(P)} \ . \tag{B.10}$$

Quite interestingly, there is a map from the set of non-zero meromorphic function f on Σ to $\mathrm{Div}(\Sigma)$, given by

$$f \mapsto (f) := \prod_{P\in\Sigma} P^{\mathrm{ord}_P f} \ . \tag{B.11}$$

Furthermore, we can define the *divisor of poles*,

$$f^{-1}(\infty) := \prod_{P\in\Sigma} P^{\max\{-\mathrm{ord}_P f, 0\}} \tag{B.12}$$

and the *divisor of zeros*

$$f^{-1}(0) := \prod_{P\in\Sigma} P^{\max\{\mathrm{ord}_P f, 0\}} \ . \tag{B.13}$$

Clearly,

$$(f) = \frac{f^{-1}(0)}{f^{-1}(\infty)} \ . \tag{B.14}$$

Let ω be a non-zero meromorphic p-form on Σ. Then one defines its divisor as

$$(\omega) := \prod_{P\in\Sigma} P^{\mathrm{ord}_P \omega} \ . \tag{B.15}$$

Divisors on hyperelliptic Riemann surfaces

Let Σ be a hyperelliptic Riemann surface of genus $\hat{g}$, and $P_1, \dots P_{2\hat{g}+2}$ points in Σ. Let further $z : \Sigma \to \mathbb{C}$ be a function on Σ, s.t. $z(P_j) \neq \infty$. Consider the following function on the Riemann surface,[1]

$$y = \sqrt{\prod_{j=1}^{2\hat{g}+2} (z - z(P_j))} \ . \tag{B.16}$$

If Q, Q' are those points on Σ for which $z(Q) = z(Q') = \infty$, i.e. $QQ' = z^{-1}(\infty)$ is the polar divisor of z, then the divisor of y is given by

$$(y) = \frac{P_1 \dots P_{2\hat{g}+2}}{Q^{\hat{g}+1} Q'^{\hat{g}+1}} \ . \tag{B.17}$$

To see this one has to introduce local coordinate patches around points $P \in \Sigma$. If P does not coincide with one of the P_i or Q, Q' local coordinates are simply $z - z(P)$. Around the points Q, Q' we have the local coordinate $1/z$, and finally around the P_i one has $\sqrt{z - z(P_i)}$.

Let R, R' be those points on Σ for which $z(R) = z(R') = 0$, i.e. $RR' = z^{-1}(0)$ is the divisor of zeros of z. The divisors for z reads

$$(z) = \frac{z^{-1}(0)}{z^{-1}(\infty)} = \frac{RR'}{QQ'} \ . \tag{B.18}$$

[1] One can show that for two points $P \neq \tilde{P}$ on Σ for which $z(P) = z(\tilde{P})$ one has $y(P) = -y(\tilde{P})$. These two branches of y are denoted y_0 and $y_1 = -y_0$ in the main text.

Furthermore, it is not hard to see that

$$(\mathrm{d}z) = \frac{P_1 \dots P_{2\hat{g}+2}}{Q^2 Q'^2} \ . \tag{B.19}$$

By simply multiplying the corresponding divisors it is then easy to see that the forms $\frac{z^k \mathrm{d}z}{y}$ are holomorphic for $k < \hat{g}$.

B.3 Relative homology and relative cohomology

A first introduction to algebraic topology can be found in [109], for a more comprehensive treatment of this beautiful subject see for example [125]. We only present the definition of relative (co-)homology, a concept that is important in a variety of problems in string theory. For example it appears naturally if one studies world-sheet instantons in the presence of D-branes, since then the world-sheet can wrap around relative cycles ending on the branes.

B.3.1 Relative homology

Let X be a triangulable manifold, Y a triangulable submanifold of X and $i : Y \to X$ its embedding. Consider chain complexes $C(X;\mathbb{Z}) := (C_j(X;\mathbb{Z}), \partial)$ and $C(Y;\mathbb{Z}) := (C_j(Y;\mathbb{Z}), \partial)$ on X and Y. For the pair (X, Y) we can define relative chain groups by

$$C_j(X, Y;\mathbb{Z}) := C_j(X;\mathbb{Z})/C_j(Y;Z) \ . \tag{B.20}$$

This means that elements of $C_j(X, Y;\mathbb{Z})$ are equivalence classes $\{c\} := c + C_j(Y;\mathbb{Z})$ and two chains c, c' in $C_j(X;\mathbb{Z})$ are in the same equivalence class if they differ only by an element c_0 of $C_j(Y;\mathbb{Z})$, $c' = c + c_0$. The relative boundary operator

$$\partial : C_j(X, Y;\mathbb{Z}) \to C_{j-1}(X, Y;\mathbb{Z}) \tag{B.21}$$

is induced by the usual boundary operator on X and Y. Indeed, two representatives c and $c' = c + c_0$ of the same equivalence class get mapped to ∂c and $\partial c'$, which satisfy $\partial c' = \partial c + \partial c_0$, and since ∂c_0 is an element of $C_{j-1}(Y;\mathbb{Z})$ their images represent the same equivalence class in $C_{j-1}(X, Y;\mathbb{Z})$.

Very importantly, the property $\partial^2 = 0$ on $C(X;\mathbb{Z})$ and $C(Y;\mathbb{Z})$ implies $\partial^2 = 0$ for the $C_j(X, Y;\mathbb{Z})$. All this defines the relative chain complex $C(X, Y;\mathbb{Z}) := (C_j(X, Y;\mathbb{Z}), \partial)$. It is natural to define the relative homology as

$$H_j(X, Y;\mathbb{Z}) := Z_j(X, Y;\mathbb{Z})/B_j(X, Y;\mathbb{Z}) \ , \tag{B.22}$$

where $Z_j(X, Y;\mathbb{Z}) := \ker(\partial) := \{\{c\} \in C_j(X, Y;\mathbb{Z}) : \partial\{c\} = 0\}$ and $B_j(X, Y;\mathbb{Z}) := \mathrm{Im}(\partial) := \{\{c\} \in C_j(X, Y;\mathbb{Z}) : \{c\} = \partial\{\hat{c}\} \text{ with } \{\hat{c}\} \in C_{j+1}(X, Y;\mathbb{Z})\} = \partial C_{j+1}(X, Y;\mathbb{Z})$. Elements of $\ker(\partial)$ are called *relative cycles* and elements of $\mathrm{Im}(\partial)$ are called *relative boundaries*. Note that the requirement that the class $\{c\}$ has no boundary, $\partial\{c\} = 0$,

or more precisely $\partial\{c\} = \{0\}$, does not necessarily mean that its representative has no boundary. It rather means that it may have a boundary but this boundary is forced to lie in Y. Note further that an element in $H_j(X,Y;\mathbb{Z})$ is an equivalence class of equivalence classes and we will denote it by $[\{c\}] := \{c\} + B_j(X,Y;\mathbb{Z})$ for $c \in C_j(X;\mathbb{Z})$ s.t. $\partial c \subset Y$.

We have the short exact sequence of chain complexes

$$0 \to C(Y;\mathbb{Z}) \xrightarrow{i} C(X;\mathbb{Z}) \xrightarrow{p} C(X,Y;\mathbb{Z}) \to 0 \tag{B.23}$$

where $i : C_j(Y;\mathbb{Z}) \to C_j(X;\mathbb{Z})$ is the obvious inclusion map and $p : C_j(X;\mathbb{Z}) \to C_j(X,Y;\mathbb{Z})$ is the projection onto the equivalence class, $p(c) = \{c\}$. Note that p is surjective, i is injective and $p \circ i = \{0\}$, which proves exactness. Every short exact sequence of chain complexes comes with a long exact sequence of homology groups. In our case

$$\ldots \to H_{j+1}(X,Y;\mathbb{Z}) \xrightarrow{\partial_*} H_j(Y;\mathbb{Z}) \xrightarrow{i_*} H_j(X;\mathbb{Z}) \xrightarrow{p_*} H_j(X,Y;\mathbb{Z}) \xrightarrow{\partial_*} H_{j-1}(Y;\mathbb{Z}) \to \ldots\ , \tag{B.24}$$

Here i_* and p_* are the homomorphisms induced from i and p in the obvious way, for example let $[c] \in H_j(X;\mathbb{Z})$ with $c \in Z_j(X;\mathbb{Z}) \subset C_j(X;\mathbb{Z})$ then $p_*([c]) := [p(c)]$. Note that $p(c) \in Z_j(X,Y;\mathbb{Z})$ and $[p(c)] = p(c) + B_j(X,Y;\mathbb{Z})$. The operator $\partial_* : H_j(X,Y;\mathbb{Z}) \to H_j(Y;\mathbb{Z})$ is defined as $[\{c\}] \mapsto [i^{-1}(\partial(p^{-1}(\{c\})))] = [\partial c]$. Here we used the fact that ∂c has to lie in Y. The symbol $[\cdot]$ denotes both equivalence classes in $H_j(X;\mathbb{Z})$ and $H_j(Y;\mathbb{Z})$.

B.3.2 Relative cohomology

Define the space of relative cohomology $H^j(X,Y;\mathbb{C})$ to be the dual space of $H_j(X,Y;\mathbb{Z})$, $H^j(X,Y;\mathbb{C}) := \mathrm{Hom}(H^j(X,Y;\mathbb{C}),\mathbb{C})$, and similarly for $H^j(X;\mathbb{C})$, $H^j(Y;\mathbb{C})$. The short exact sequence (B.23) comes with a dual exact sequence

$$0 \to \mathrm{Hom}(C(X,Y;\mathbb{Z}),\mathbb{C}) \xrightarrow{\tilde{p}} \mathrm{Hom}(C(X;\mathbb{Z}),\mathbb{C}) \xrightarrow{\tilde{i}} \mathrm{Hom}(C(Y;\mathbb{Z}),\mathbb{C}) \to 0\ . \tag{B.25}$$

Here we need the definition of the dual homomorphism for a general chain mapping[2] $f : C(X;\mathbb{Z}) \to C(Y;\mathbb{Z})$. Let $\phi \in \mathrm{Hom}(C(Y;\mathbb{Z}),\mathbb{C})$, then $\tilde{f}(\phi) := \phi \circ f$.
The corresponding long exact sequence reads

$$\ldots \to H^{j-1}(X;\mathbb{C}) \xrightarrow{i^*} H^{j-1}(Y;\mathbb{C}) \xrightarrow{\mathrm{d}^*} H^j(X,Y;\mathbb{C}) \xrightarrow{p^*} H^j(X;\mathbb{C}) \xrightarrow{i^*} H^j(Y;\mathbb{C}) \to \ldots\ , \tag{B.26}$$

where $i^* := \tilde{i}_*$, $p^* := \tilde{p}_*$. For example let $[\Theta] \in H^j(X;\mathbb{C})$. Then $i^*([\Theta]) = \tilde{i}_*([\Theta]) = [\tilde{i}(\Theta)]$. d^* acts on $[\theta] \in H^{j-1}(Y;\mathbb{C})$ as $\mathrm{d}^*([\theta]) := [\tilde{p}^{-1}(\mathrm{d}(\tilde{i}^{-1}(\theta)))]$, and $\mathrm{d} := \tilde{\partial}$ is the *coboundary operator*.

[2] A chain mapping $f : C(X;\mathbb{Z}) \to C(Y;\mathbb{Z})$ is a family of homomorphisms $f_j : C_j(X;\mathbb{Z}) \to C_j(Y;\mathbb{Z})$ which satisfy $\partial \circ f_j = f_{j-1} \circ \partial$.

So far we started from simplicial complexes, introduced homology groups on triangulable spaces and defined the cohomology groups as their duals. On the other hand, there is a natural set of cohomology spaces on a differential manifold, very familiar to physicist, namely the de Rham cohomology groups. In fact, they encode exactly the same topological information, as we have for any triangulable differential manifold X that

$$H^j_{deRham}(X;\mathbb{C}) \cong H^j_{simplicial}(X;\mathbb{C}) \ . \tag{B.27}$$

So we can interpret the spaces appearing in the long exact sequence (B.26) as de Rham groups, and the maps i^* and p^* are the pullbacks corresponding to i and p. Furthermore, the coboundary operator $\mathrm{d} \equiv \tilde{\partial}$ is nothing but the exterior derivative.

In fact, on a differentiable manifold X with a closed submanifold Y the relative cohomology groups $H^j(X,Y;\mathbb{C})$ can be defined from forms on X [88]. Let $\Omega^j(X,Y;\mathbb{C})$ be the j-forms on X that vanish on Y, i.e.

$$\Omega^j(X,Y;\mathbb{C}) := \ker(\Omega^j(X;\mathbb{C}) \xrightarrow{i^*} \Omega^j(Y)) \tag{B.28}$$

where i^* is the pullback corresponding to the inclusion $i : Y \to X$. Then it is natural to define

$$\begin{aligned} Z^j(X,Y;\mathbb{C}) &:= \{\Theta \in \Omega^j(X,Y;\mathbb{C}) : \mathrm{d}\Theta = 0\} \ , \\ B^j(X,Y;\mathbb{C}) &:= \{\Theta \in \Omega^j(X,Y;\mathbb{C}) : \Theta = \mathrm{d}\eta \text{ for } \eta \in \Omega^{j-1}(X,Y;\mathbb{C})\} \ , \\ H^j(X,Y;\mathbb{C}) &:= Z^j(X,Y;\mathbb{C})/B^j(X,Y;\mathbb{C}) \ . \end{aligned} \tag{B.29}$$

As for the cohomology spaces we have

$$H^j_{deRham}(X,Y;\mathbb{C}) \cong H^j_{simplicial}(X,Y;\mathbb{C}) \ . \tag{B.30}$$

There is a pairing

$$\langle .,. \rangle : H_j(X,Y;\mathbb{Z}) \times H^j(X,Y;\mathbb{C}) \to \mathbb{C} \tag{B.31}$$

for $[\Gamma] \in H_j(X,Y;\mathbb{Z})$ and $[\Theta] \in H^j(X,Y;\mathbb{C})$, defined as

$$\langle [\Gamma],[\Theta] \rangle := \int_\Gamma \Theta \ . \tag{B.32}$$

Of course, this pairing is independent of the chosen representative in the two equivalence classes. For $\Gamma' := \Gamma + \partial\hat{\Gamma} + \Gamma_0$ with $\Gamma_0 \subset Y$ this follows immediately from the fact that Θ is closed and vanishes on Y. For $\Theta' := \Theta + \mathrm{d}\Lambda$ this follows from $\int_\Gamma \mathrm{d}\Lambda = \int_{\partial\Gamma} \Lambda = 0$, as Λ vanishes on Y.

This definition can be extended to the space of forms

$$\tilde{\Omega}^j(X,Y;\mathbb{C}) := \{\Theta \in \Omega^j(X;\mathbb{C}) : i^*\Theta = \mathrm{d}\theta\} \ . \tag{B.33}$$

Then one first defines the space of equivalence classes

$$\hat{\Omega}^j(X,Y;\mathbb{C}) := \tilde{\Omega}^j(X,Y;\mathbb{C})/\mathrm{d}\Omega^{j-1}(X;\mathbb{C}) \ . \tag{B.34}$$

which is clearly isomorphic to $\Omega^j(X,Y;\mathbb{C})$. Note that an element of this space is an equivalence class $\{\Theta\} := \Theta + \mathrm{d}\Omega^{j-1}(X;\mathbb{C})$. Then one continues as before, namely one defines

$$\begin{aligned}
\hat{Z}^j(X,Y;\mathbb{C}) &:= \{\{\Theta\} \in \hat{\Omega}^j(X,Y;\mathbb{C}) : \mathrm{d}\{\Theta\} = 0\} \;, \\
\hat{B}^j(X,Y;\mathbb{C}) &:= \{\{\Theta\} \in \hat{\Omega}^j(X,Y;\mathbb{C}) : \{\Theta\} = \mathrm{d}\{\eta\} \text{ for } \{\eta\} \in \hat{\Omega}^{j-1}(X,Y;\mathbb{C})\} \;, \\
\hat{H}^j(X,Y;\mathbb{C}) &:= \hat{Z}^j(X,Y;\mathbb{C})/\hat{B}^j(X,Y;\mathbb{C}) \;.
\end{aligned} \tag{B.35}$$

Obviously, $\hat{H}^j(X,Y;\mathbb{C}) \cong H^j(X,Y;\mathbb{C})$. Element of $\hat{H}^j(X,Y;\mathbb{C})$ are equivalence classes of equivalence classes and we denote them by $[\{\Theta\}]$. The natural pairing in this case is given by

$$\begin{aligned}
\langle .,. \rangle \quad &: \quad H_j(X,Y;\mathbb{Z}) \times \hat{H}^j(X,Y;\mathbb{C}) \to \mathbb{C} \\
([\Gamma],[\{\Theta\}]) \quad &\mapsto \quad \langle [\Gamma],[\{\Theta\}] \rangle := \int_\Gamma \Theta - \int_\Gamma \mathrm{d}\theta \quad \text{if } i^*\Theta = \mathrm{d}\theta \;,
\end{aligned} \tag{B.36}$$

which again is independent of the representative.

The group $\hat{H}^j(X,Y;\mathbb{C})$ can be characterised in yet another way. Note that a representative Θ of $\hat{Z}(X,Y;\mathbb{C})$ has to be closed $\mathrm{d}\Theta = 0$ and it must pull back to an exact form on Y, $i^*\Theta = \mathrm{d}\theta$. This motives us to define an exterior derivative

$$\begin{aligned}
\mathrm{d} : \Omega^j(X;\mathbb{C}) \times \Omega^{j-1}(Y;\mathbb{C}) \quad &\to \quad \Omega^{j+1}(X;\mathbb{C}) \times \Omega^j(Y;\mathbb{C}) \\
(\Theta,\theta) \quad &\mapsto \quad \mathrm{d}(\Theta,\theta) := (\mathrm{d}\Theta, i^*\Theta - \mathrm{d}\theta) \;.
\end{aligned} \tag{B.37}$$

It is easy to check that $\mathrm{d}^2 = 0$. If we define

$$\mathcal{Z}(X,Y;\mathbb{C}) := \{(\Theta,\theta) \in \Omega^j(X;\mathbb{C}) \times \Omega^{j-1}(Y;\mathbb{C}) : \mathrm{d}(\Theta,\theta) = 0\} \tag{B.38}$$

we find

$$\mathcal{Z}(X,Y;\mathbb{C}) \cong \hat{Z}(X,Y;\mathbb{C}) \;. \tag{B.39}$$

Note further that $\hat{B}^j(X,Y;\mathbb{C}) \cong B^j(X;\mathbb{C})$, so we require that the representative Θ and $\Theta + \mathrm{d}\Lambda$ are equivalent. But $i^*(\Theta + \mathrm{d}\Lambda) = \mathrm{d}(\theta + i^*\Lambda)$ so θ has to be equivalent to $\theta + i^*\Lambda$. This can be captured by

$$(\Theta,\theta) \sim (\Theta,\theta) + \mathrm{d}(\Lambda,\lambda) \;, \tag{B.40}$$

and with

$$\mathcal{B}(X,Y;\mathbb{C}) := \{\mathrm{d}(\Lambda,\lambda) : (\Lambda,\lambda) \in \Omega^j(X;\mathbb{C}) \times \Omega^{j-1}(Y;\mathbb{C})\} \tag{B.41}$$

we have

$$\hat{H}(X,Y;\mathbb{C}) \cong \mathcal{Z}(X,Y;\mathbb{C})/\mathcal{B}(X,Y;\mathbb{C}) \;. \tag{B.42}$$

B.4 Index theorems

It turns out that anomalies are closely related to the index of differential operators. A famous theorem found by Atiyah and Singer tells us how to determine the index

of these operators from topological quantities. In this section we collect important index theory results which are needed to calculate the anomalies. [109] gives a rather elementary introduction to index theorems. Their relation to anomalies is explained in [11] and [12].

Theorem B.4 *(*Atiyah-Singer index theorem*)*
Let M be a manifold of even dimension, $d = 2n$, G a Lie group, $P(M, G)$ the principal bundle of G over M and let E the associated vector bundle with $k := \dim(E)$. Let A be the gauge potential corresponding to a connection on E and let $S_\pm$ be the positive and negative chirality part of the spin bundle. Define the Dirac operators $D_\pm : S_\pm \otimes E \to S_\mp \otimes E$ by

$$D_\pm := i\Gamma^M \left(\partial_M + \frac{1}{4}\omega_{MAB}\Gamma^{AB} + A_M\right) P_\pm \ . \tag{B.43}$$

Then $\mathrm{ind}(D_+)$ *with*

$$\mathrm{ind}(D_+) := \dim(\ker D_+) - \dim(\ker D_-) \tag{B.44}$$

is given by

$$\mathrm{ind}(D_+) \ = \ \int_M [\mathrm{ch}(F)\hat{A}(M)]_{\mathrm{vol}} \ , \tag{B.45}$$

$$\hat{A}(M) \ := \ \prod_{j=1}^{n} \frac{x_j/2}{\sinh(x_j/2)} \ , \tag{B.46}$$

$$\mathrm{ch}(F) \ := \ \mathrm{tr} \ \exp\left(\frac{iF}{2\pi}\right) \ . \tag{B.47}$$

The x_j are defined as

$$p(E) := \det\left(1 + \frac{R}{2\pi}\right) = \prod_{j=1}^{[n/2]}(1 + x_j^2) = 1 + p_1 + p_2 + \dots \ , \tag{B.48}$$

where $p(E)$ is the *total Pontrjagin class* of the bundle E. The x_j are nothing but the skew eigenvalues of $R/2\pi$,

$$\frac{R}{2\pi} = \begin{pmatrix} 0 & x_1 & 0 & 0 & \dots \\ -x_1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & x_2 & \dots \\ 0 & 0 & -x_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \ . \tag{B.49}$$

$\hat{A}(M)$ is known as the *Dirac genus* and $\mathrm{ch}(F)$ is the *total Chern character*. The subscript vol means that one has to extract the form whose degree equals the dimension of M.

To read off the volume form both $\hat{A}(M)$ and $\mathrm{ch}(F)$ need to be expanded. We get [11], [12]

$$\begin{aligned}
\hat{A}(M) &= 1+\frac{1}{(4\pi)^2}\frac{1}{12}\mathrm{tr}R^2+\frac{1}{(4\pi)^4}\left[\frac{1}{288}(\mathrm{tr}R^2)^2+\frac{1}{360}\mathrm{tr}R^4\right] \\
&+\frac{1}{(4\pi)^6}\left[\frac{1}{10368}(\mathrm{tr}R^2)^3+\frac{1}{4320}\mathrm{tr}R^2\mathrm{tr}R^4+\frac{1}{5670}\mathrm{tr}R^6\right] \\
&+\frac{1}{(4\pi)^8}\left[\frac{1}{497664}(\mathrm{tr}R^2)^4+\frac{1}{103680}(\mathrm{tr}R^2)^2\mathrm{tr}R^4+\right. \\
&\left.+\frac{1}{68040}\mathrm{tr}R^2\mathrm{tr}R^6+\frac{1}{259200}(\mathrm{tr}R^4)^2+\frac{1}{75600}\mathrm{tr}R^8\right]+\ldots\ , \qquad \text{(B.50)} \\
\mathrm{ch}(F) &:= \mathrm{tr}\exp\left(\frac{iF}{2\pi}\right)=k+\frac{i}{2\pi}\mathrm{tr}F+\frac{i^2}{2(2\pi)^2}\mathrm{tr}F^2+\ldots+\frac{i^s}{s!(2\pi)^s}\mathrm{tr}F^s+\ldots\ . \qquad \text{(B.51)}
\end{aligned}$$

From these formulae we can determine the index of the Dirac operator on arbitrary manifolds, e.g. for $d=4$ we get

$$\mathrm{ind}(D_+)=\frac{1}{(2\pi)^2}\int_M\left(\frac{i^2}{2}\mathrm{tr}F^2+\frac{k}{48}\mathrm{tr}R^2\right)\ . \qquad \text{(B.52)}$$

The Dirac operator (B.43) is not the only operator we need in order to calculate anomalies. We also need the analogue of (B.45) for spin-3/2 fields which is given by [11], [12]

$$\begin{aligned}
\mathrm{ind}(D_{3/2}) &= \int_M[\hat{A}(M)(\mathrm{tr}\exp(iR/2\pi)-1)\mathrm{ch}(F)]_{\mathrm{vol}} \\
&= \int_M[\hat{A}(M)(\mathrm{tr}(\exp(iR/2\pi)-\mathbb{1})+d-1)\mathrm{ch}(F)]_{\mathrm{vol}}\ . \qquad \text{(B.53)}
\end{aligned}$$

Explicitly,

$$\begin{aligned}
\hat{A}(M)\mathrm{tr}\,(\exp(R/2\pi)-\mathbb{1}) &= -\frac{1}{(4\pi)^2}\ 2\ \mathrm{tr}R^2 \\
&+\frac{1}{(4\pi)^4}\left[-\frac{1}{6}(\mathrm{tr}R^2)^2+\frac{2}{3}\mathrm{tr}R^4\right] \\
&+\frac{1}{(4\pi)^6}\left[-\frac{1}{144}(\mathrm{tr}R^2)^3+\frac{1}{20}\mathrm{tr}R^2\mathrm{tr}R^4-\frac{4}{45}\mathrm{tr}R^6\right] \\
&+\frac{1}{(4\pi)^8}\left[-\frac{1}{5184}(\mathrm{tr}R^2)^4+\frac{1}{540}(\mathrm{tr}R^2)^2\mathrm{tr}R^4-\right. \\
&\left.-\frac{22}{2835}\mathrm{tr}R^2\mathrm{tr}R^6+\frac{1}{540}(\mathrm{tr}R^4)^2+\frac{2}{315}\mathrm{tr}R^8\right] \\
&+\ldots \qquad \text{(B.54)}
\end{aligned}$$

Finally, in $4k+2$ dimensions there are anomalies related to forms with (anti-)self-dual field strength. The relevant index is given by [11], [12]

$$\mathrm{ind}(D_A) = \frac{1}{4}\int_M [L(M)]_{2n} \ , \tag{B.55}$$

where the subscript A stands for anti-symmetric tensor. $L(M)$ is known as the *Hirzebruch L-polynomial* and is defined as

$$L(M) := \prod_{j=1}^{n} \frac{x_j/2}{\tanh(x_j/2)} \ . \tag{B.56}$$

For reference we present the expansion

$$\begin{aligned} L(M) \ &= \ 1 - \frac{1}{(2\pi)^2}\frac{1}{6}\mathrm{tr}R^2 + \frac{1}{(2\pi)^4}\left[\frac{1}{72}(\mathrm{tr}R^2)^2 - \frac{7}{180}\mathrm{tr}R^4\right] \\ &+ \frac{1}{(2\pi)^6}\left[-\frac{1}{1296}(\mathrm{tr}R^2)^3 + \frac{7}{1080}\mathrm{tr}R^2\mathrm{tr}R^4 - \frac{31}{2835}\mathrm{tr}R^6\right] \\ &+ \frac{1}{(2\pi)^8}\left[\frac{1}{31104}(\mathrm{tr}R^2)^4 - \frac{7}{12960}(\mathrm{tr}R^2)^2\mathrm{tr}R^4 + \right. \\ &\left. + \frac{31}{17010}\mathrm{tr}R^2\mathrm{tr}R^6 + \frac{49}{64800}(\mathrm{tr}R^4)^2 - \frac{127}{37800}\mathrm{tr}R^8\right] + \ldots \ . \end{aligned} \tag{B.57}$$

Appendix C

Special Geometry and Picard-Fuchs Equations

In section 4.2 we saw that the moduli space of a compact Calabi-Yau manifold carries a Kähler metric and that the Kähler potential can be calculated from some holomorphic function $\mathcal{F}$, which itself can be obtained from geometric integrals. In this appendix we will show that this moduli space is actually an example of a mathematical structure known as special Kähler manifold. We also explain how a set of differential equations, the so called Picard Fuchs equations arise. In the case of compact Calabi-Yau manifolds these are differential equations for the period integrals of Ω. This is interesting, since for general Calabi-Yau manifolds it may well be simpler to solve the differential equation than to explicitly calculate the period integral. During my thesis some progress on the extension of the concept of special geometry to local Calabi-Yau manifolds has been made [P5]. Unfortunately, the analogue of the Picard-Fuchs equations is still unknown for these manifolds. A first attempt to set up a rigorous, coordinate free framework for special Kähler manifolds was made in [127]. Various properties of special geometry and its relation to the moduli spaces of Riemann surfaces and Calabi-Yau manifolds, as well as to Picard-Fuchs equations, were studied in [56], [36], [35]. For a detailed analysis of the various possible definitions see [38].

C.1 (Local) Special geometry

We start with the definition of a (local) special Kähler manifold, which is modelled after the moduli space of complex structures of compact Calabi-Yau manifolds, and which should be contrasted with rigid special Kähler manifolds, that occur in the context of Riemann surfaces.

Hodge manifolds

Consider a complex n-dimensional Kähler manifold with coordinates $z^i, \bar{z}^{\bar{\jmath}}$ and metric

$$g_{i\bar{\jmath}} = \partial_i \bar{\partial}_{\bar{\jmath}} K(z, \bar{z}) \ , \tag{C.1}$$

where the real function $K(z,\bar z)$ is the Kähler potential, $i,j,\dots,\bar\imath,\bar\jmath,\dots \in \{1,\dots,n\}$. The Christoffel symbols and the Riemann tensor are calculated from the standard formulae

$$\Gamma^i_{jk} = g^{i\bar l}\partial_j g_{k\bar l}\ , \quad R^i{}_{j\bar k l} = \partial_{\bar k}\Gamma^i_{jl}\ , \tag{C.2}$$

with $g_{i\bar\jmath}g^{k\bar\jmath} = \delta_i^k$, $g_{i\bar k}g^{i\bar\jmath} = \delta_{\bar k}^{\bar\jmath}$, and the Kähler form is defined as

$$\omega := i g_{i\bar\jmath}\mathrm{d}z^i \wedge \mathrm{d}\bar z^{\bar\jmath}\ . \tag{C.3}$$

Introduce the one-form

$$Q := -\frac{1}{4}\partial K + \frac{1}{4}\bar\partial K\ . \tag{C.4}$$

Then under a Kähler transformation,

$$\tilde K(\tilde z(z), \bar{\tilde z}(\bar z)) = K(z,\bar z) + f(z) + \bar f(\bar z)\ , \tag{C.5}$$

g is invariant and

$$\tilde Q = Q - \frac{1}{4}\partial f + \frac{1}{4}\bar\partial\bar f\ . \tag{C.6}$$

Clearly

$$\omega = 2i\mathrm{d}Q\ . \tag{C.7}$$

Definition C.1 A *Hodge manifold* is a Kähler manifold carrying a $U(1)$ line bundle $\mathcal{L}$, s.t. Q is the connection of $\mathcal{L}$. Then the first Chern-class of $\mathcal{L}$ is given in terms of the Kähler class, $2c_1(\mathcal{L}) = \frac{1}{2\pi}[\omega]$. Such a manifold is sometimes also called a *Kähler manifold of restricted type.*[1]

On a Hodge manifold a section of $\mathcal{L}$ of Kähler weight $(q,\bar q)$ transforms as

$$\tilde\psi(\tilde z(z), \bar{\tilde z}(\bar z)) = \psi(z,\bar z)e^{\frac{q}{4}f(z)}e^{-\frac{\bar q}{4}\bar f(\bar z)} \tag{C.8}$$

when going from one patch to another. For these we introduce the Kähler covariant derivatives

$$\mathcal{D}\psi := \left(\partial - \frac{q}{4}(\partial K)\right)\psi \quad , \quad \bar{\mathcal{D}}\psi := \left(\bar\partial + \frac{\bar q}{4}(\bar\partial K)\right)\psi\ , \tag{C.9}$$

where $\partial := \mathrm{d}z^i\partial_i$, $\bar\partial := \mathrm{d}\bar z^{\bar\imath}\bar\partial_{\bar\imath} := \mathrm{d}\bar z^{\bar\imath}\frac{\partial}{\partial\bar z^{\bar\imath}}$. A Kähler covariantly holomorphic section, i.e. one satisfying $\bar D\psi = 0$ is related to a purely holomorphic section ψ_{hol} by

$$\psi_{hol} = e^{\frac{\bar q}{4}K}\psi\ , \tag{C.10}$$

since then $\bar\partial\psi_{hol} = 0$. The Kähler covariant derivative can be extended to tensors Φ on the Hodge manifold $\mathcal{M}$ as

$$\mathcal{D}\Phi = \left(\nabla - \frac{q}{4}(\partial K)\right)\Phi \quad , \quad \bar{\mathcal{D}}\Phi = \left(\bar\nabla + \frac{\bar q}{4}(\bar\partial K)\right)\Phi\ , \tag{C.11}$$

[1] Following [38] we define the Hodge manifold to have a Kähler form which, when multiplied by $\frac{1}{2\pi}$, is of even integer cohomology. In the mathematical literature it is usually defined as a manifold with Kähler form of integer homology, $c_1(\mathcal{L}) = \frac{1}{2\pi}[\omega]$.

where ∇ is the standard covariant derivative.

Special Kähler manifolds

Next we want to define the notion of a special Kähler manifold, following the conventions of [38]. There are in fact three different definitions that are useful and which are all equivalent.

Definition C.2 A *special Kähler manifold* $\mathcal{M}$ is a complex n-dimensional Hodge manifold with the following properties:

- On every chart there are complex projective coordinate functions $X^I(z)$, $I \in \{0, \ldots, n\}$, and a holomorphic function $\mathcal{F}(X^I)$, which is homogeneous of second degree, i.e. $2\mathcal{F} = X^I \mathcal{F}_I := X^I \frac{\partial}{\partial X^I}\mathcal{F}$, s.t. the Kähler potential is given as

$$K(z,\bar{z}) = -\log\left[i\bar{X}^I \frac{\partial}{\partial X^I}\mathcal{F}(X^I) - iX^I \frac{\partial}{\partial \bar{X}^I}\bar{\mathcal{F}}(\bar{X}^I)\right] . \tag{C.12}$$

- On overlaps of charts, U_α, U_β, the corresponding functions are connected by

$$\begin{pmatrix} X \\ \partial\mathcal{F} \end{pmatrix}_{(\alpha)} = e^{-f_{(\alpha\beta)}} M_{(\alpha\beta)} \begin{pmatrix} X \\ \partial\mathcal{F} \end{pmatrix}_{(\beta)} , \tag{C.13}$$

  where $f_{(\alpha\beta)}$ is holomorphic on $U_\alpha \cap U_\beta$ and $M_{(\alpha\beta)} \in Sp(2n+2, \mathbb{R})$.

- On the overlap of three charts, $U_\alpha, U_\beta, U_\gamma$, the transition functions satisfy the cocycle condition,

$$\begin{aligned} e^{f_{(\alpha\beta)}+f_{(\beta\gamma)}+f_{(\gamma\alpha)}} &= 1 , \\ M_{(\alpha\beta)}M_{(\beta\gamma)}M_{(\gamma\alpha)} &= 1 . \end{aligned} \tag{C.14}$$

Definition C.3 A special Kähler manifold $\mathcal{M}$ is a complex n-dimensional Hodge manifold, s.t.

- $\exists$ a holomorphic $Sp(2n+2, \mathbb{R})$ vector bundle $\mathcal{H}$ over $\mathcal{M}$ and a holomorphic section $v(z)$ of $\mathcal{L} \otimes \mathcal{H}$, s.t. the Kähler form is

$$\omega = -i\partial\bar{\partial}\log(i\langle \bar{v}, v\rangle) , \tag{C.15}$$

  where $\mathcal{L}$ denotes the holomorphic line bundle over $\mathcal{M}$ that appeared in the definition of a Hodge manifold, and $\langle v, w\rangle := v^\tau \mho w$ is the symplectic product on $\mathcal{H}$;

- furthermore, this section satisfies

$$\langle v, \partial_i v\rangle = 0 . \tag{C.16}$$

Definition C.4 Let $\mathcal{M}$ be a complex n-dimensional manifold, and let $V(z,\bar{z})$ be a $2n+2$-component vector defined on each chart, with transition function

$$V_{(\alpha)} = e^{-\frac{1}{2}f_{(\alpha\beta)}} e^{\frac{1}{2}\bar{f}_{(\alpha\beta)}} M_{(\alpha\beta)} V_{(\beta)} \ , \tag{C.17}$$

where $f_{\alpha\beta}$ is a holomorphic function on the overlap and $M_{(\alpha\beta)}$ a constant $Sp(2n+2,\mathbb{R})$ matrix. The transition functions have to satisfy the cocycle condition. Take a $U(1)$ connection of the form $\kappa_i \mathrm{d}z^i - \bar{\kappa}_{\bar{i}} \mathrm{d}\bar{z}^{\bar{i}}$, under which $\bar{V}$ has opposite charge as V. Define

$$\begin{aligned} U_i := \mathcal{D}_i V &:= (\partial_i + \kappa_i) V \ , \\ \bar{\mathcal{D}}_{\bar{i}} V &:= (\bar{\partial}_{\bar{i}} - \bar{\kappa}_{\bar{i}}) V \ , \\ \bar{U}_{\bar{i}} := \bar{\mathcal{D}}_{\bar{i}} \bar{V} &:= (\bar{\partial}_{\bar{i}} + \bar{\kappa}_{\bar{i}}) \bar{V} \ , \\ \mathcal{D}_i \bar{V} &:= (\partial_i - \kappa_i) \bar{V} \ , \end{aligned} \tag{C.18}$$

and impose

$$\begin{aligned} \langle V, \bar{V} \rangle &= i \ , \\ \bar{\mathcal{D}}_{\bar{i}} V &= 0 \ , \\ \langle V, U_i \rangle &= 0 \ , \\ \mathcal{D}_{[i} U_{j]} &= 0 \ . \end{aligned} \tag{C.19}$$

Finally define

$$g_{i\bar{j}} := i \langle U_i, \bar{U}_{\bar{j}} \rangle \ . \tag{C.20}$$

If this is a positive metric on $\mathcal{M}$ then $\mathcal{M}$ is a special Kähler manifold.

The first definition is modelled after the structure found in the scalar sector of four-dimensional supergravity theories. It is useful for calculations as everything is given in terms of local coordinates. However, it somehow blurs the coordinate independence of the concept of special geometry. This is made explicit in the second definition which, although somewhat abstract, captures the structure of the space. The third definition, finally, can be used to show that the moduli space of complex structures of a Calabi-Yau manifold is a special Kähler manifold. The proof of the equivalence of these three definitions can be found in [38].

Consequences of special geometry

In order to work out some of the implications of special geometry we start from a special Kähler manifold that satisfies all the properties of definition C.3. Clearly, (C.15) is equivalent to $K = -\log(i\langle \bar{v}, v \rangle)$ and from the transformation property of K we deduce that v is a field of weight $(-4,0)$ and $\bar{v}$ of weight (0,4). In other words they transform as

$$\tilde{v} = e^{-f} M v \ , \quad \tilde{\bar{v}} = e^{-\bar{f}} M \bar{v} \ , \tag{C.21}$$

from one local patch to another. Here $M \in Sp(2n+2,\mathbb{R})$. The corresponding derivatives are

$$\begin{aligned} \mathcal{D}_i v &= (\partial_i + (\partial_i K))\, v \ , \\ \bar{\mathcal{D}}_{\bar{i}} v &= \bar{\partial}_{\bar{i}} v \ , \end{aligned} \tag{C.22}$$

and their complex conjugates. Let us define

$$\begin{aligned} u_i &:= \mathcal{D}_i v \ , \\ \bar{u}_{\bar{i}} &:= \bar{\mathcal{D}}_{\bar{i}} \bar{v} \ , \end{aligned} \tag{C.23}$$

i.e. u_i is a section of $T \otimes \mathcal{L} \otimes \mathcal{H}$ of weight $(-4,0)$ and $\bar{u}_{\bar{i}}$ a section of $\bar{T} \otimes \bar{\mathcal{L}} \otimes \mathcal{H}$ of weight $(0,4)$. Here we denoted $T := T^{(1,0)}(\mathcal{M})$, $\bar{T} := T^{(0,1)}(\mathcal{M})$. Then the following relations hold, together with their complex conjugates

$$\bar{\mathcal{D}}_{\bar{i}} v = 0 \ , \tag{C.24}$$

$$\mathcal{D}_i \bar{u}_{\bar{j}} = [\mathcal{D}_i, \bar{\mathcal{D}}_{\bar{j}}]\bar{v} = g_{i\bar{j}} \bar{v} \ , \tag{C.25}$$

$$[\mathcal{D}_i, \mathcal{D}_j] = 0 \ , \tag{C.26}$$

$$\mathcal{D}_{[i} u_{j]} = 0 \ , \tag{C.27}$$

$$\langle v, v \rangle = 0 \ , \tag{C.28}$$

$$\langle v, \bar{v} \rangle = i e^{-K} \ , \tag{C.29}$$

$$\langle v, u_i \rangle = 0 \ , \tag{C.30}$$

$$\langle u_i, u_j \rangle = 0 \ , \tag{C.31}$$

$$\langle v, \bar{u}_{\bar{i}} \rangle = 0 \ , \tag{C.32}$$

$$\langle u_i, \bar{u}_{\bar{j}} \rangle = -i g_{i\bar{j}} e^{-K} \ . \tag{C.33}$$

(C.24) holds as v is holomorphic by definition, (C.25), (C.26) and (C.27) can be found by spelling out the covariant derivatives, (C.28) is trivial. (C.29) is (C.15). Note that (C.16) can be written as $\langle v, \mathcal{D}_i v \rangle = 0$ because of the antisymmetry of $\langle ., . \rangle$, this gives (C.30). Taking the covariant derivative and antisymmetrising gives $\langle \mathcal{D}_i v, \mathcal{D}_j v \rangle = 0$ and thus (C.31). Taking the covariant derivative of (C.29) leads to (C.32). Taking another covariant derivative leads to the last relation.
Next we define the important quantity

$$C_{ijk} := -i e^K \langle \mathcal{D}_i \mathcal{D}_j v, \mathcal{D}_k v \rangle = -i e^K \langle \mathcal{D}_i u_j, u_k \rangle \ , \tag{C.34}$$

which has weight $(-4,-4)$ and satisfies

$$C_{ijk} = C_{(ijk)} \ , \tag{C.35}$$

$$\bar{\mathcal{D}}_{\bar{l}} C_{ijk} = 0 \ , \tag{C.36}$$

$$\mathcal{D}_{[i} C_{j]kl} = 0 \ , \tag{C.37}$$

$$\mathcal{D}_i u_j = C_{ijk} \bar{u}^k \ , \tag{C.38}$$

where $\bar{u}^j := g^{j\bar{k}} \bar{u}_{\bar{k}}$. The first two relations can be proven readily from the equations that have been derived so far. The third relation follows from $\langle u_i, [\mathcal{D}_j, \mathcal{D}_k] u_l \rangle = 0$. In

order to prove (C.38) one expands $\mathcal{D}_i u_j$ as[2] $\mathcal{D}_i u_j = a_{ij} v + b_{ij}^k u_k + c_{ij}^{\bar{k}} \bar{u}_{\bar{k}} + d_{ij}\bar{v}$, and determines the coefficients by taking symplectic products with the basis vectors. The result is that ${c_{ij}}^{\bar{k}} = {C_{ij}}^{\bar{k}}$ and all other coefficients vanish. Considering $\langle u_i, [\mathcal{D}_k, \bar{\mathcal{D}}_{\bar{l}}]\bar{u}_{\bar{j}}\rangle$ we are then led to a formula for the Riemann tensor,

$$R_{\bar{i}j\bar{k}l} = g_{j\bar{k}} g_{l\bar{i}} + g_{l\bar{k}} g_{j\bar{i}} - C_{jlm} {\bar{C}_{\bar{k}\bar{i}}}{}^{m}. \tag{C.39}$$

Here $\bar{C}_{\bar{i}\bar{j}\bar{k}}$ is the complex conjugate of C_{ijk} and ${C_{\bar{k}\bar{i}}}^{m} := g^{m\bar{m}} C_{\bar{k}\bar{i}\bar{m}}$.

Matrix formulation

So far we collected these properties in a rather unsystematic and not very illuminating way. In order to improve the situation we define the $(2n+2)\times(2n+2)$-matrix

$$\mathcal{U} := \begin{pmatrix} v^\tau \\ {u_j}^\tau \\ (\bar{u}_{\bar{k}})^\tau \\ \bar{v}^\tau \end{pmatrix}, \tag{C.40}$$

which satisfies

$$\mathcal{U}\mho\mathcal{U}^\tau = ie^{-K}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -g_{j\bar{k}} & 0 \\ 0 & g_{\bar{k}j} & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{C.41}$$

as can be seen from (C.28)-(C.33). In other words one defines the bundle $E := (\mathcal{L} \oplus (T\otimes\mathcal{L}) \oplus (\bar{T}\otimes\bar{\mathcal{L}}) \oplus \bar{\mathcal{L}}) \otimes \mathcal{H} = \mathrm{span}(v) \oplus \mathrm{span}(u_i) \oplus \mathrm{span}(\bar{u}_{\bar{i}}) \oplus \mathrm{span}(\bar{v})$ and $\mathcal{U}$ is a section of E. Let us first study the transformation of $\mathcal{U}$ under a change of coordinate patches. We find

$$\tilde{\mathcal{U}} = S^{-1}\mathcal{U}M^\tau \tag{C.42}$$

where $M \in Sp(2n+2,\mathbb{R})$,

$$S^{-1} = \begin{pmatrix} e^{-f} & 0 & 0 & 0 \\ 0 & e^{-f}\xi^{-1} & 0 & 0 \\ 0 & 0 & e^{-\bar{f}}\bar{\xi}^{-1} & 0 \\ 0 & 0 & 0 & e^{-\bar{f}} \end{pmatrix} \tag{C.43}$$

and $\xi := \xi^i_j := \frac{\partial \tilde{z}^i}{\partial z^j}$, $\bar{\xi} := \bar{\xi}^{\bar{i}}_{\bar{j}} := \frac{\partial \bar{\tilde{z}}^{\bar{i}}}{\partial \bar{z}^{\bar{j}}}$. Using (C.23), (C.38), (C.25) and (C.24) one finds that on a special Kähler manifold the following matrix equations are satisfied[3],

[2] To understand this it is useful to look at the example of the complex structure moduli space $\mathcal{M}_{cs}$ of a Calabi-Yau manifold, which is of course the example we have in mind during the entire discussion. As we will see in more detail below, v is given by the period vector of Ω and the u_i have to be interpreted as the period vectors of a basis of the $(2,1)$-forms. Then it is clear that $\bar{u}_{\bar{i}}$ should be understood as period vectors of a basis of $(1,2)$-forms and finally $\bar{v}$ is given by the period vector of $\bar{\Omega}$. The derivative of a $(2,1)$-form can be expressed as a linear combination of the basis elements, which is the expansion of $\mathcal{D}_i u_j$ in terms of v, u_i, $\bar{u}_{\bar{i}}$, $\bar{v}$, which now is obvious as these form a basis of three-forms.

[3] Note that the matrices $\bar{\mathbb{C}}$ and $\bar{\mathbb{A}}$ are *not* the complex conjugates of $\mathbb{C}$ and $\mathbb{A}$.

$$
\begin{aligned}
\mathcal{D}_i \mathcal{U} &= \mathbb{C}_i \mathcal{U} \ , \\
\bar{\mathcal{D}}_{\bar{i}} \mathcal{U} &= \bar{\mathbb{C}}_{\bar{i}} \mathcal{U} \ ,
\end{aligned} \tag{C.44}
$$

with

$$
\mathbb{C}_i = \begin{pmatrix} 0 & \delta_i^j & 0 & 0 \\ 0 & 0 & C_{ij}{}^{\bar{k}} & 0 \\ 0 & 0 & 0 & g_{i\bar{k}} \\ 0 & 0 & 0 & 0 \end{pmatrix} , \quad \bar{\mathbb{C}}_{\bar{i}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ g_{\bar{i}j} & 0 & 0 & 0 \\ 0 & \bar{C}_{\bar{i}\bar{j}}{}^{k} & 0 & 0 \\ 0 & 0 & \delta_{\bar{i}}^{\bar{k}} & 0 \end{pmatrix} . \tag{C.45}
$$

Here δ_i^j is a row vector of dimension n with a 1 at position i and zeros otherwise, $g_{i\bar{k}}$ is a column vector of dimension n with entry $g_{i\bar{k}}$ at position $\bar{k}$, and $C_{ij}{}^{\bar{k}}$ is symbolic for the $n \times n$-matrix C_i with matrix elements $(C_i)_j{}^{\bar{k}} = C_{ij}{}^{\bar{k}}$. The entries of $\bar{\mathbb{C}}_i$ are to be understood similarly. These matrices satisfy $[\mathbb{C}_i, \mathbb{C}_j] = 0$ and $\mathbb{C}_i\mathbb{C}_j\mathbb{C}_k\mathbb{C}_l = 0$. It will be useful to rephrase these equations in a slightly different form. We introduce the operator matrix $\mathbb{D} := \mathbb{D}_i \mathrm{d}z^i$,

$$
\begin{aligned}
\mathbb{D}_i \mathcal{U} &:= (\partial_i + \mathbb{A}_i)\mathcal{U} := (\mathcal{D}_i - \mathbb{C}_i)\mathcal{U} = (\partial_i + \mathbf{\Gamma}_i - \mathbb{C}_i)\mathcal{U} = 0 \ , \\
\bar{\mathbb{D}}_{\bar{i}} \mathcal{U} &:= (\bar{\partial}_{\bar{i}} + \bar{\mathbb{A}}_i)\mathcal{U} := (\bar{\mathcal{D}}_{\bar{i}} - \bar{\mathbb{C}}_{\bar{i}})\mathcal{U} = (\bar{\partial}_{\bar{i}} + \bar{\mathbf{\Gamma}}_{\bar{i}} - \bar{\mathbb{C}}_{\bar{i}})\mathcal{U} = 0 \ ,
\end{aligned} \tag{C.46}
$$

where

$$
\mathbf{\Gamma}_i := \begin{pmatrix} \partial_i K & 0 & 0 & 0 \\ 0 & \delta_j^k \partial_i K - \Gamma_{ij}^k & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \quad \bar{\mathbf{\Gamma}}_{\bar{i}} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \delta_{\bar{j}}^{\bar{k}} \bar{\partial}_{\bar{i}} K - \bar{\Gamma}_{\bar{i}\bar{j}}^{\bar{k}} & 0 \\ 0 & 0 & 0 & \bar{\partial}_{\bar{i}} K \end{pmatrix} , \tag{C.47}
$$

and therefore

$$
\mathbb{A}_i := \begin{pmatrix} \partial_i K & \delta_i^j & 0 & 0 \\ 0 & \delta_j^k \partial_i K - \Gamma_{ij}^k & C_{ij}{}^{\bar{k}} & 0 \\ 0 & 0 & 0 & g_{i\bar{k}} \\ 0 & 0 & 0 & 0 \end{pmatrix} , \quad \bar{\mathbb{A}}_{\bar{i}} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ g_{\bar{i}j} & 0 & 0 & 0 \\ 0 & \bar{C}_{\bar{i}\bar{j}}{}^{k} & \delta_{\bar{j}}^{\bar{k}} \bar{\partial}_{\bar{i}} K - \bar{\Gamma}_{\bar{i}\bar{j}}^{\bar{k}} & 0 \\ 0 & 0 & \delta_{\bar{i}}^{\bar{j}} & \bar{\partial}_{\bar{i}} K \end{pmatrix} . \tag{C.48}
$$

Let us see how $\mathbb{D}$ transforms when we change patches. From the transformation properties of $\mathcal{D}$ and $\mathbb{C}$ we find

$$
\begin{aligned}
\tilde{\mathbb{A}} &= S^{-1} \mathbb{A} S + S^{-1} \mathrm{d}S \ , \\
\tilde{\bar{\mathbb{A}}} &= S^{-1} \bar{\mathbb{A}} S + S^{-1} \mathrm{d}S \ ,
\end{aligned} \tag{C.49}
$$

for $\mathbb{A} := \mathbb{A}_i \mathrm{d}z^i$, so $\mathbb{A}$ is a connection. One easily verifies that

$$
[\mathbb{D}_i, \mathbb{D}_j] = [\bar{\mathbb{D}}_{\bar{i}}, \bar{\mathbb{D}}_{\bar{j}}] = 0 \ . \tag{C.50}
$$

Relation (C.39) for the Riemann tensor gives

$$
[\mathbb{D}_i, \bar{\mathbb{D}}_{\bar{j}}]\mathcal{U} = 0 \ . \tag{C.51}
$$

All this implies that $\mathbb{D}$ is flat on the bundle E.

Special coordinates and holomorphic connections
We see that an interesting structure emerges once we write our equations in matrix form. To push this further we need to apply the equivalence of definition C.3 and C.2. In particular, we want to make use of the fact that locally, i.e. on a given chart on $\mathcal{M}$, we can write

$$v = \begin{pmatrix} X^I \\ \frac{\partial \mathcal{F}}{\partial X^J} \end{pmatrix} , \tag{C.52}$$

where $X^I = X^I(z)$ and $\mathcal{F} = \mathcal{F}(X(z))$ are holomorphic. Furthermore, the X^I can be taken to be projective coordinates on our chart. There is a preferred coordinate system given by

$$t^a(z) := \frac{X^a}{X^0}(z) , \quad a \in \{1, \ldots, n\} , \tag{C.53}$$

and the t^a are known as *special coordinates*. To see that this coordinate system really is useful, let us reconsider the covariant derivative $\mathcal{D}_i$ on the special Kähler manifold $\mathcal{M}$, which contains $\Gamma(z,\bar{z})$, $K_i(z,\bar{z}) := \partial_i K(z,\bar{z})$. If we make use of special coordinates these split into a holomorphic and a non-holomorphic piece. We start from $\Gamma^k_{ij}(z,\bar{z})$, set $e_i{}^a := \frac{\partial t^a(z)}{\partial z^i}$, which does depend on z but not on $\bar{z}$, and write

$$\begin{aligned} \Gamma^k_{ij} &= g^{k\bar{l}} \partial_i g_{j\bar{l}} = g^{a\bar{b}} (e^{-1})_a{}^k (e^{-1})_{\bar{b}}{}^{\bar{l}} e_i{}^c \partial_c (e_j{}^d e_{\bar{l}}{}^{\bar{f}} g_{d\bar{f}}) \\ &= e_i{}^c (\partial_c e_j{}^d)(e^{-1})_d{}^k + e_i{}^c e_j{}^d (\partial_c g_{d\bar{f}}) g^{a\bar{f}} (e^{-1})_a{}^k \\ &=: \hat{\Gamma}^k_{ij}(z) + T^k_{ij}(z,\bar{z}) . \end{aligned} \tag{C.54}$$

Note that $\hat{\Gamma}^k_{ij}$ transforms as a connection under a holomorphic coordinate transformation $z \to \tilde{z}(z)$, whereas T^k_{ij} transforms as a tensor. Similarly from (C.12) one finds that

$$\begin{aligned} &K_i(z,\bar{z}) = -\partial_i \log(X^0(z)) \\ &-\partial_i \log \left[i \frac{\bar{X}^0(\bar{z})}{X^0(z)} \left(\frac{\partial}{\partial X^0} \mathcal{F}(X^0(z), X^a(z)) + \bar{t}^a(\bar{z}) \frac{\partial}{\partial X^a} \mathcal{F}(X^0(z), X^a(z)) \right) \right. \\ &\left. -i \frac{\partial}{\partial \bar{X}^0} \bar{\mathcal{F}}(\bar{X}^0(\bar{z}), \bar{X}^a(\bar{z})) - i t^a(z) \frac{\partial}{\partial \bar{X}^a} \bar{\mathcal{F}}(\bar{X}^0(\bar{z}), \bar{X}^a(\bar{z})) \right] \\ &=: \hat{K}_i(z) + \mathcal{K}_i(z,\bar{z}) . \end{aligned} \tag{C.55}$$

Clearly, $\mathcal{K}_i(z,\bar{z})$ is invariant under Kähler transformations and $\hat{K}_i(z)$ transforms as $\hat{K}_i(z) \to \hat{K}_i(z) + \partial_i f(z)$, which is precisely the transformation law of $K_i(z,\bar{z})$. Thus, the transformation properties of Γ and K_i are carried entirely by the holomorphic parts and one can define *holomorphic* covariant derivatives for any tensor Φ on $\mathcal{M}$ of weight $(q,\bar{q})$,

$$\begin{aligned} \hat{\mathcal{D}}_i \Phi &:= \left(\hat{\nabla}_i - \frac{q}{4} (\partial_i \hat{K}) \right) \Phi , \\ \bar{\hat{\mathcal{D}}}_{\bar{i}} \Phi &:= \left(\bar{\hat{\nabla}}_{\bar{i}} + \frac{\bar{q}}{4} (\bar{\partial}_{\bar{i}} \hat{K}) \right) \Phi , \end{aligned} \tag{C.56}$$

where $\hat{\nabla}$ now only contains the holomorphic connection $\hat{\Gamma}$. A most important fact is that $\hat{\Gamma}$ is flat, i.e. the corresponding curvature tensor vanishes,

$$\hat{R}^k{}_{lij} := \partial_l \hat{\Gamma}^k_{ij} - \partial_i \hat{\Gamma}^k_{lj} + \hat{\Gamma}^m_{lj} \hat{\Gamma}^k{}_{mi} - \hat{\Gamma}^m_{lj} \hat{\Gamma}^k{}_{mi} = 0 \ , \tag{C.57}$$

which can be seen readily from the explicit form of $\hat{\Gamma}$. Next we take η_{ab} to be a constant, invertible symmetric matrix and define

$$\hat{g}_{ij} := e_i{}^a e_j{}^b \eta_{ab} \ . \tag{C.58}$$

It is not difficult to show that the Levi-Civita connection of $\hat{g}_{ij}$ is nothing but $\hat{\Gamma}$. If we take $z^i = t^a$ one finds that

$$e_i{}^a = \delta^a_i \quad , \quad \hat{\Gamma}^k_{ij} = 0 \quad , \quad \hat{g}_{ij} = \eta_{ij} \ . \tag{C.59}$$

So the holomorphic part of the connection can be set to zero if we work in special coordinates. Special or flat coordinates are a preferred coordinate system on a chart of our base manifold $\mathcal{M}$. But on such a chart v is only defined up to a transformation $\tilde{v} = e^{-f} M v$. This tells us that we can choose a gauge for v such that $X^0 = 1$. This is, of course, the gauge in which the holomorphic part of the Kähler connection vanishes as well, $\hat{K}_i = 0$.

Solving the Picard-Fuchs equation

The fact that the connections on a special Kähler manifold can be split into a holomorphic connection part and a non-holomorphic tensor part is highly non-trivial. In the following we present one very important consequence of this fact. Consider once again the system

$$\mathbb{D}\mathcal{U} = 0 \quad , \quad \bar{\mathbb{D}}\mathcal{U} = 0 \tag{C.60}$$

which holds on any special Kähler manifold. Now suppose we know of a manifold that it is special Kähler, but we do not know the holomorphic section v. Then we can understand (C.60) as a differential system for $\mathcal{U}$ and therefore for v. These differential equations are known as the *Picard-Fuchs equations*. As to solve this system consider the transformation

$$\begin{aligned} \mathcal{U} \ &\rightarrow \ \mathcal{W} := R^{-1}\mathcal{U} \ , \\ \mathbb{A} \ &\rightarrow \ \hat{\mathbb{A}} := R^{-1}(\mathbb{A} + \partial) R \ , \\ \bar{\mathbb{A}} \ &\rightarrow \ \hat{\bar{\mathbb{A}}} := R^{-1}(\bar{\mathbb{A}} + \bar{\partial}) R \ , \end{aligned} \tag{C.61}$$

where

$$R^{-1}(z,\bar{z}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ * & \mathbb{1} & 0 & 0 \\ * & * & \mathbb{1} & 0 \\ * & * & * & 1 \end{pmatrix} \ . \tag{C.62}$$

Note that for such a matrix R^{-1} the matrix R will be lower diagonal with all diagonal elements equal to one, as well. Clearly, this transformation leaves v invariant and

$$\begin{aligned} \mathbb{D}\mathcal{U} = 0 \rightarrow \hat{\mathbb{D}}\mathcal{W} = 0 \ , \\ \bar{\mathbb{D}}\mathcal{U} = 0 \rightarrow \hat{\bar{\mathbb{D}}}\mathcal{W} = 0 \ , \end{aligned} \tag{C.63}$$

with $\hat{\mathbb{D}} := \partial + \hat{\mathbb{A}}$ and $\hat{\bar{\mathbb{D}}} := \bar{\partial} + \hat{\bar{\mathbb{A}}}$. The crucial point is that the solution v that we are after does not change under this transformation, i.e. we might as well study the system $\hat{\mathbb{D}}\mathcal{W} = 0$, $\hat{\bar{\mathbb{D}}}\mathcal{W} = 0$. Next we note that $R^{-1}(z, \bar{z})$ does not have to be holomorphic. In fact, since the curvature of $\bar{\mathbb{A}}$ vanishes we can go into a system where $\hat{\bar{\mathbb{A}}} = 0$ and therefore $\bar{\mathbb{A}} = R\bar{\partial}R^{-1}$. In this gauge we have

$$\bar{\partial}\mathcal{W} = 0 \ , \tag{C.64}$$

and from (C.51) one finds

$$\bar{\partial}\hat{\mathbb{A}} = 0 \ , \tag{C.65}$$

which tells us that the non-holomorphic parts of the connection $\hat{\mathbb{A}}$ vanish and that the matrices $\mathbb{C}_i$ are holomorphic. In other words we have $\hat{\mathbb{D}}_i = \hat{\mathcal{D}}_i - \mathbb{C}_i$, where $\hat{\mathcal{D}}_i$ is the holomorphic covariant derivative of Eq. (C.56) including $\hat{K}_i(z)$ and $\hat{\Gamma}^k_{ij}$ only and we are interested in the solutions of the *holomorphic* system

$$\hat{\mathbb{D}}\mathcal{W} = 0 \ . \tag{C.66}$$

Note that in this system we still have holomorphic coordinate transformations as a residual symmetry. Furthermore, if we plug (C.52) in the definition of C_{ijk}, we find in the holomorphic system

$$C_{ijk} = -ie^{\hat{K}} \left[(\hat{\mathcal{D}}_i\hat{\mathcal{D}}_j X^I)\hat{\mathcal{D}}_k\mathcal{F}_I - (\hat{\mathcal{D}}_i\hat{\mathcal{D}}_j\mathcal{F}_I)\hat{\mathcal{D}}_k X^I \right] \ . \tag{C.67}$$

In special coordinates this reduces to

$$C_{abc} = i\partial_a\partial_b\partial_c\mathcal{F} \ . \tag{C.68}$$

The strategy is now to solve the system in special coordinates first, and to find the general solution from a holomorphic coordinate transformation afterwards. Let us then choose special coordinates on a chart of $\mathcal{M}$, together with the choice $X^0 = 1$, and let

$$\mathcal{W} =: \begin{pmatrix} v^\tau \\ v_a^\tau \\ \tilde{v}_b^\tau \\ (v^0)^\tau \end{pmatrix} \ . \tag{C.69}$$

In special coordinates the equation $\hat{\mathbb{D}}_a\mathcal{W} = 0$ reads $(\partial_a - \mathbb{C}_a)\mathcal{W} = 0$ with

$$\mathbb{C}_a = \begin{pmatrix} 0 & \delta_a^b & 0 & 0 \\ 0 & 0 & C_{ab}{}^c & 0 \\ 0 & 0 & 0 & \eta_{ac} \\ 0 & 0 & 0 & 0 \end{pmatrix} \ . \tag{C.70}$$

or

$$\begin{aligned} \partial_a v &= v_a \ , \\ \partial_a v_b &= C_{ab}{}^c \tilde{v}_c \ , \\ \partial_a \tilde{v}_c &= \eta_{ac} v^0 \ , \\ \partial_a v^0 &= 0 \ . \end{aligned} \tag{C.71}$$

Recalling (C.68) the solution can be found to be

$$\begin{pmatrix} v^\tau \\ v_b^\tau \\ \tilde{v}_c^\tau \\ (v^0)^\tau \end{pmatrix} = \begin{pmatrix} 1 & t^d & \partial_d \mathcal{F} & 2\mathcal{F} - t^e \partial_e \mathcal{F} \\ 0 & \delta_b^d & \partial_b \partial_d \mathcal{F} & \partial_b \mathcal{F} - t^e \partial_e \partial_b \mathcal{F} \\ 0 & 0 & -i\eta_{cd} & it^e \eta_{ec} \\ 0 & 0 & 0 & i \end{pmatrix} . \tag{C.72}$$

Of course, the solution of $\hat{\mathbb{D}}\mathcal{W} = 0$ in general (holomorphic) coordinates can then be obtained from a (holomorphic) coordinate transformation of (C.72).

Special geometry and Calabi-Yau manifolds

It was shown in section 4.2 that the moduli space of complex structures of a compact Calabi-Yau manifold is Kähler with Kähler potential $K = -\log\left(i \int \Omega \wedge \bar{\Omega}\right)$. Let

$$V := e^{\frac{K}{2}} \int_{\Gamma_i^{(3)}} \Omega^{(3,0)} \ , \tag{C.73}$$

where $\{\Gamma_i^{(3)}\} = \{\Gamma_{\alpha^I}, \Gamma_{\beta_J}\}$ is the set of all three-cycles in X. These cycles are only defined up to a symplectic transformation and Ω is defined up to a transformation $\Omega \to e^{f(z)}\Omega$, see Eq. (4.38). Therefore V is defined only up to a transformation

$$V \to \tilde{V} = e^{-\frac{f}{2}} e^{\frac{\bar{f}}{2}} M V \ , \tag{C.74}$$

with $M \in Sp(2n+2; \mathbb{Z})$ and $f(z)$ holomorphic. Next we set

$$U_i \ := \ \mathcal{D}_i V := \left(\partial_i + \frac{1}{2} K_i\right) V = \left(\partial_i + \frac{1}{2}(\partial_i K)\right) V \ , \tag{C.75}$$

$$\bar{\mathcal{D}}_{\bar{i}} V \ := \ \left(\bar{\partial}_{\bar{i}} - \frac{1}{2}(\bar{\partial}_{\bar{i}} K)\right) V \ . \tag{C.76}$$

For vectors $V = \int_{\Gamma_i^{(3)}} \Xi$ and $W = \int_{\Gamma_i^{(3)}} \Sigma$ with Ξ, Σ three-forms on X there is a natural symplectic product, $\langle V, W \rangle := -\int_X \Xi \wedge \Sigma$. Then it is easy to verify that (C.19) holds and $i\langle U_i, \bar{U}_{\bar{j}} \rangle = -\frac{\int_X \chi_i \wedge \bar{\chi}_{\bar{j}}}{\int_X \Omega \wedge \bar{\Omega}}$ is indeed nothing but $G^{(CS)}_{i\bar{j}}$. So we find that all the requirements of definition C.4 are satisfied and $\mathcal{M}_{cs}$ is a special Kähler manifold.

In the context of Calabi-Yau manifolds the projective coordinates are given by the integrals of the unique holomorphic three-form Ω over the Γ_{α^I}-cycles and the

holomorphic function $\mathcal{F}$ of definition C.2 is nothing but the prepotential constructed from integrals of Ω over the Γ_{β_J}-cycles,

$$X^I = \int_{\Gamma_{\alpha^I}} \Omega \quad , \quad \mathcal{F}_I = \int_{\Gamma_{\beta_I}} \Omega \ . \tag{C.77}$$

In other words the vector v of definition C.3 is given by the period vector of Ω and the symplectic bundle is the Hodge bundle $\mathcal{H}$. This is actually the main reason why the structure of special geometry is so important for physicist. The integrals of Ω over three cycles is interesting since it calculates the prepotential (and therefore the physically very interesting quantity C_{ijk}). On the other hand, using mirror symmetry a period integral on one manifold can tell us something about the instanton structure on another manifold. Unfortunately the integrals often cannot be calculated explicitly. However, in some cases it is possible to solve the ordinary linear differential Picard-Fuchs equation for $\int \Omega$, which therefore gives an alternative way of extracting interesting quantities. Indeed, the differential equations that one can derive for $\int \Omega$ by hand agree with the Picard-Fuchs equations of special geometry. Finally we note that the matrix $\mathcal{U}$ is nothing but the period matrix of the Calabi-Yau,

$$\mathcal{U} = \begin{pmatrix} \int \Omega^{(3,0)} \\ \int \Omega_\alpha^{(2,1)} \\ \int \Omega_\alpha^{(1,2)} \\ \int \Omega^{(0,3)} \end{pmatrix} \ , \tag{C.78}$$

which can be brought into holomorphic form by a (non-holomorphic) gauge transformation.

C.2 Rigid special geometry

The quantum field theories we are interested in are generated from local Calabi-Yau manifolds, rather than from compact Calabi-Yau manifolds. We saw already that the integrals of Ω over (relative) three-cycles maps to integrals on a Riemann surface. It turns out that the moduli space of Riemann surfaces carries a structure which is very similar to the special geometry of Calabi-Yau manifolds, and which is known as rigid special geometry.

Rigid special Kähler manifolds

Definition C.5 A complex n-dimensional Kähler manifold is said to be *rigid special Kähler* if it satisfies:

- On every chart there are n independent holomorphic functions $X^i(z)$, $i \in \{1, \ldots, n\}$ and a holomorphic function $\mathcal{F}(X)$, s.t.

$$K(z, \bar{z}) = i \left(X^i \frac{\partial}{\partial \bar{X}^i} \bar{\mathcal{F}}(\bar{X}^i) - \bar{X}^i \frac{\partial}{\partial X^i} \mathcal{F}(X^i) \right) . \tag{C.79}$$

- On overlaps of charts there are transition functions of the form

$$\begin{pmatrix} X \\ \partial F \end{pmatrix}_{(\alpha)} = e^{ic_{\alpha\beta}} M_{\alpha\beta} \begin{pmatrix} X \\ \partial F \end{pmatrix} + b_{\alpha\beta}, \tag{C.80}$$

  with $c_{\alpha\beta} \in \mathbb{R}$, $M_{\alpha\beta} \in Sp(2n, \mathbb{R})$, $b_{\alpha\beta} \in \mathbb{C}^{2n}$.

- The transition functions satisfy the cocycle condition on overlaps of three charts.

Definition C.6 A rigid special Kähler manifold is a Kähler manifold $\mathcal{M}$ with the following properties:

- There exists a $U(1) \times ISp(2n, \mathbb{R})$ vector bundle[4] over $\mathcal{M}$ with constant transition functions, in the sense of (E.8), i.e. with a complex inhomogeneous piece. This bundle should have a holomorphic section V, s.t. the Kähler form is

$$\omega = -\partial\bar{\partial}\langle V, \bar{V}\rangle \ . \tag{C.81}$$

- We have $\langle \partial_i V, \partial_j V\rangle = 0$.

Definition C.7 A rigid special manifold is a complex n-dimensional Kähler manifold with on each chart $2n$ closed holomorphic 1-forms $U_i \mathrm{d}z^i$,

$$\bar{\partial}_i U_j = 0 \ , \quad \partial_{[i} U_{j]} = 0 \ , \tag{C.82}$$

with the following properties:

- $\langle U_i, U_j\rangle = 0$.

- The Kähler metric is

$$g_{ij} = i\langle U_i, \bar{U}_j\rangle \ . \tag{C.83}$$

- The transition functions read

$$U_{i,(\alpha)} \mathrm{d}z^i_{(\alpha)} = e^{ic_{\alpha\beta}} M_{\alpha\beta} U_{i,(\beta)} \mathrm{d}z^i_{(\beta)} \tag{C.84}$$

  with $c_{\alpha\beta} \in \mathbb{R}$, $M_{\alpha\beta} \in Sp(2n, \mathbb{R})$.

- The cocycle condition holds on the overlap of three charts.

[4] This is a vector bundle with transition function $V_{(\alpha)} = e^{ic_{\alpha\beta}} M_{(\alpha\beta)} V + b_{\alpha\beta}$, with $c_{\alpha\beta} \in \mathbb{R}$, $M_{\alpha\beta} \in Sp(2n, \mathbb{R})$, $b_{\alpha\beta} \in \mathbb{C}^{2n}$.

For the proof of the equivalence of these definitions see [38].

Matrix formulation and Picard Fuchs equations
As before we want to work out some of the consequences of these definitions. Take the U_i from Def. C.7 and define

$$\mathcal{V} := \begin{pmatrix} U_i^\tau \\ \bar{U}_{\bar{j}}^\tau \end{pmatrix} . \tag{C.85}$$

Equations (C.82) and (C.83) can then be written in matrix form

$$\mathcal{V}\mho\mathcal{V}^\tau = \begin{pmatrix} 0 & -ig_{i\bar{j}} \\ ig_{j\bar{i}} & 0 \end{pmatrix} . \tag{C.86}$$

Next we define the $(2n \times 2n)$-matrix

$$\mathbb{A}_i := -(\partial_i \mathcal{V})\mathcal{V}^{-1} , \tag{C.87}$$

which has the structure

$$\mathbb{A}_i = -\begin{pmatrix} G_{ij}{}^k & C_{ij}{}^{\bar{k}} \\ 0 & 0 \end{pmatrix} , \tag{C.88}$$

where $G_{ij}{}^k = G_{(i,j)}{}^k$ and $C_{ij}{}^{\bar{k}} = C_{(i,j)}{}^{\bar{k}}$, as can be seen from Eq. (C.82). Then one finds

$$\begin{aligned} \mathbb{A}_i \begin{pmatrix} 0 & -ig_{k\bar{l}} \\ ig_{l\bar{k}} & 0 \end{pmatrix} &= -(\partial_i \mathcal{V})\Omega\mathcal{V}^\tau \\ &= -\partial_i \begin{pmatrix} 0 & -ig_{j\bar{l}} \\ ig_{l\bar{j}} & 0 \end{pmatrix} + \mathcal{V}\Omega\partial_i\mathcal{V}^\tau , \end{aligned} \tag{C.89}$$

but on the other hand

$$\mathbb{A}_i \begin{pmatrix} 0 & -ig_{k\bar{l}} \\ ig_{l\bar{k}} & 0 \end{pmatrix} = \begin{pmatrix} -iC_{(i,j)l} & iG_{(i,j)\bar{l}} \\ 0 & 0 \end{pmatrix} , \tag{C.90}$$

and therefore, taking the second line of (C.89) minus the transpose of the first,

$$\begin{pmatrix} iC_{(i,j)l} - iC_{(i,l)j} & -iG_{(i,j)\bar{l}} \\ iG_{(i,l)\bar{j}} & 0 \end{pmatrix} = \partial_i \begin{pmatrix} 0 & -ig_{j\bar{l}} \\ ig_{l\bar{j}} & 0 \end{pmatrix} . \tag{C.91}$$

We deduce that C is symmetric in all its indices and that $\partial_{[i}g_{j]\bar{l}} = 0$, i.e. it is Kähler. Hence, we can define the Levi-Civita connection and its covariant derivative

$$\Gamma^k_{ij} = g^{k\bar{l}}\partial_j g_{i\bar{l}} \quad , \quad \nabla_i U_j := \partial_i U_j - \Gamma^k_{ij} U_k \quad , \quad \nabla_i \bar{U}_{\bar{j}} := \partial_i \bar{U}_{\bar{j}} . \tag{C.92}$$

Clearly, $G_{ij}{}^k = \Gamma^k_{ij}$ and we find

$$(\partial_i + \mathbb{A}_i)\mathcal{V} = (\nabla_i - \mathbb{C}_i)\mathcal{V} = 0 \quad , \quad (\bar{\partial}_{\bar{i}} + \bar{\mathbb{A}}_{\bar{i}})\mathcal{V} = (\bar{\nabla}_{\bar{i}} - \bar{\mathbb{C}}_{\bar{i}})\mathcal{V} = 0 , \tag{C.93}$$

with

$$\bar{\mathbb{A}}_{\bar{i}} := \begin{pmatrix} 0 & 0 \\ -\bar{C}_{\bar{i}\bar{j}}{}^k & -\bar{\Gamma}^{\bar{k}}_{\bar{i}\bar{j}} \end{pmatrix} \quad , \quad \mathbb{C}_i := \begin{pmatrix} 0 & C_{ij}{}^{\bar{k}} \\ 0 & 0 \end{pmatrix} \quad , \quad \bar{\mathbb{C}}_i := \begin{pmatrix} 0 & 0 \\ \bar{C}_{\bar{i}\bar{j}}{}^k & 0 \end{pmatrix} \tag{C.94}$$

and symmetric C_{ijk}. From $[\nabla_i, \nabla_j]\mathcal{V} = 0$ we deduce that $\nabla_{[i}\mathbb{C}_{j]} = 0$ or

$$C_{ijk} = \nabla_i \nabla_j \nabla_k \mathcal{S} \tag{C.95}$$

for some function $\mathcal{S}$. Acting with $[\nabla_i, \bar{\nabla}_{\bar{j}}]$ on $\mathcal{V}$ gives the identity

$$R_{i\bar{j}k\bar{l}} = -C_{ikm}\bar{C}_{\bar{j}\bar{l}\bar{m}}g^{m\bar{m}} \ . \tag{C.96}$$

Finally we define $\mathbb{D}_i := \nabla_i - \mathbb{C}_i$ and $\bar{\mathbb{D}}_{\bar{i}} := (\bar{\nabla}_i - \bar{\mathbb{C}}_i)$ with the properties $[\mathbb{D}_i, \mathbb{D}_j] = 0$, $[\bar{\mathbb{D}}_{\bar{i}}, \bar{\mathbb{D}}_{\bar{j}}] = 0$ and $[\mathbb{D}_i, \bar{\mathbb{D}}_{\bar{j}}]\mathcal{V} = 0$, so $\mathbb{D}$ is a flat connection.

Now we use the equivalence of the three definitions and define the *special coordinates* to be the holomorphic functions

$$t^a(z) := X^i(z) \ . \tag{C.97}$$

As above the Christoffel symbol splits into a tensor part and a holomorphic connection part,

$$\Gamma^k_{ij}(z,\bar{z}) = \hat{\Gamma}^k_{ij}(z) + T^k_{ij}(z,\bar{z}), \tag{C.98}$$

and once again $\hat{\Gamma}$ is a flat connection, $R(\hat{\Gamma}) = 0$, that vanishes if we use special coordinates. We again define a holomorphic covariant derivative, $\hat{\nabla}$, that contains only the holomorphic part of the connection. Its commutator gives the curvature of $\hat{\Gamma}$ and therefore vanishes. Next we transform

$$\begin{aligned} \mathcal{V} &\rightarrow \mathcal{X} := S^{-1}\mathcal{V} \ , \\ \mathbb{A} &\rightarrow \hat{\mathbb{A}} := S^{-1}(\mathbb{A} + \partial)S \ , \\ \bar{\mathbb{A}} &\rightarrow \bar{\hat{\mathbb{A}}} := S^{-1}(\bar{\mathbb{A}} + \bar{\partial})S \ , \end{aligned} \tag{C.99}$$

where S is chosen such that $\bar{\hat{\mathbb{A}}} = 0$. Then we are left with

$$(\hat{\nabla}_\alpha - \hat{\mathbb{C}})\mathcal{X} = 0 \quad , \quad \bar{\partial}_{\bar{\alpha}}\mathcal{X} = 0 \ . \tag{C.100}$$

In special coordinates $\hat{\Gamma}$ vanishes and this reads

$$\partial_a \mathcal{X} = \begin{pmatrix} 0 & C_{ab}{}^c \\ 0 & 0 \end{pmatrix} \mathcal{X} \ . \tag{C.101}$$

Using the equivalence of the three definitions we find that

$$U_i = \partial_i \begin{pmatrix} X^j \\ \partial_j \mathcal{F} \end{pmatrix} \ . \tag{C.102}$$

Then, multiplying $\partial_a U_b = C_{ab}{}^d \bar{U}_d$ by $U_c^\tau \mho$ we find that in special coordinates

$$C_{abc} = i\partial_a\partial_b\partial_c\mathcal{F} \ , \tag{C.103}$$

which leads to the solution

$$\mathcal{X} = \begin{pmatrix} \delta^d_b & \partial_b\partial_d\mathcal{F} \\ 0 & -i\eta_{cd} \end{pmatrix} \ . \tag{C.104}$$

Rigid special geometry and Riemann surfaces

For a given Riemann surface Σ it is natural to try to construct a rigid special Kähler manifold using the $\hat{g}$ holomorphic forms λ_i. However, the moduli space of a Riemann surface has dimension $3\hat{g}-3$ for $\hat{g}>1$, whereas we can only construct a $\hat{g}$-dimensional rigid special Kähler manifold from our forms. That means that unless $\hat{g}=1$ the special manifold will be a submanifold of the moduli space of Σ.

Let then $\mathcal{W}$ be a family of genus $\hat{g}$ Riemann surfaces which are parameterised by only $\hat{g}$ complex moduli and let λ_i be the set of holomorphic one-forms on Σ. Then we identify

$$U_i = \begin{pmatrix} \int_{\alpha^j} \lambda_i \\ \int_{\beta_k} \lambda_i \end{pmatrix} . \tag{C.105}$$

$\bar{\partial}_{\bar{j}} U_i = 0$ tells us that the periods should depend holomorphically on the moduli. Using Riemann's second relation one can show that all requirements of definition (C.7) are satisfied. (E.g. $\langle U_i, U_j \rangle = 0$ follows immediately from the symmetry of the period matrix; Riemann's second relation gives the positivity of the metric.) The condition $\partial_{[j} U_{i]} = 0$ has to be checked for the particular example one is considering. On compact Riemann surfaces it reduces to $\partial_{[i} \lambda_{j]} = (\mathrm{d}\eta)_{ij}$. If the right-hand side is zero then locally there exists a meromorphic one-form λ whose derivatives give the holomorphic one-forms. Then, using

$$U_i = \partial_i \begin{pmatrix} X^j \\ \partial_k \mathcal{F} \end{pmatrix} , \tag{C.106}$$

we identify

$$\begin{pmatrix} X^i \\ \partial_j \mathcal{F} \end{pmatrix} = \begin{pmatrix} \int_{\alpha^i} \lambda \\ \int_{\beta_j} \lambda \end{pmatrix} . \tag{C.107}$$

This implies that, similarly to the case of the Calabi-Yau manifold, the holomorphic function X^i and the prepotential $\mathcal{F}$ can be calculated from geometric integrals on the Riemann surface. However, in contrast to the Calabi-Yau space, the form λ now is meromorphic and not holomorphic.

Appendix D

Topological String Theory

One of the central building blocks of the web of theories sketched in Fig. 2.3 is the B-type topological string. Indeed, although the relation between effective superpotentials and matrix models can also be proven using field theory results only [42], [26], the string theory derivation of this relation leads to the insight that many seemingly different theories are actually very much related, and the topological string lies at the heart of these observations. The reason for this central position of the topological string are a number of its properties. It has been known for a long time [20] that the topological string computes certain physical amplitudes of type II string theories compactified on Calabi-Yau manifolds. Furthermore, it turns out that the string field theory of the open B-type topological string on the simple manifolds X_{res} is an extremely simple gauge theory, namely a holomorphic matrix model [43], as is reviewed in chapter 7. Finally, we already saw in the introduction that the gauge theory/string theory duality can be made precise if the string theory is topological. Here we review the definition of the topological string, together with some of its elementary properties. Since topological string theory is a vast and quickly developing subject, this review will be far from complete. The reader is referred to the book of Hori et. al. [81], for more details and references. For a review of more recent developments see for example [111]. Here we follow the pedagogical introduction of [133].

D.1 Cohomological field theories

Before we embark on defining the topological string let us define a *cohomological field theory*. It has the following properties:

- It contains a fermionic symmetry operator Q that squares to zero,

$$Q^2 = 0 \ . \tag{D.1}$$

  Then the *physical operators* of the theory are defined to be those that are Q-closed,

$$\{Q, \mathcal{O}_i\} = 0 \ . \tag{D.2}$$

- The vacuum is Q-symmetric, i.e. $Q|0\rangle = 0$. This implies the equivalence $\mathcal{O}_i \sim \mathcal{O}_i + \{Q, \Lambda\}$, as can be seen from $\langle \mathcal{O}_{i_1} \ldots \mathcal{O}_{i_k} \{Q, \Lambda\} \mathcal{O}_{i_{k+2}} \ldots \rangle = 0$, where we used (D.2).

- The energy-momentum tensor is Q-exact,

$$T_{\alpha\beta} \equiv \frac{\delta S}{\delta h^{\alpha\beta}} = \{Q, G_{\alpha\beta}\} \ . \tag{D.3}$$

  This property implies that the correlation functions do not depend on the metric,

$$\begin{aligned} \frac{\delta}{\delta h^{\alpha\beta}} \langle \mathcal{O}_{i_1} \ldots \mathcal{O}_{i_n} \rangle &= \frac{\delta}{\delta h^{\alpha\beta}} \int D\phi \ \mathcal{O}_{i_1} \ldots \mathcal{O}_{i_n} e^{iS[\phi]} \\ &= i \int D\phi \ \mathcal{O}_{i_1} \ldots \mathcal{O}_{i_n} e^{iS[\phi]} \frac{\delta S}{\delta h^{\alpha\beta}} \\ &= i \langle \mathcal{O}_{i_1} \ldots \mathcal{O}_{i_n} \{Q, G_{\alpha\beta}\} \rangle = 0 \ . \end{aligned} \tag{D.4}$$

  Here we assumed that the $\mathcal{O}_i$ do not depend on the metric.

The condition (D.3) is trivially satisfied if the Lagrangian is Q-exact,

$$L = \{Q, V\} \ , \tag{D.5}$$

for some operator V. For such a Lagrangian one can actually calculate the correlation functions *exactly* in the classical limit. To see this note that

$$\frac{\partial}{\partial \hbar} \langle \mathcal{O}_{i_1} \ldots \mathcal{O}_{i_n} \rangle = \frac{\partial}{\partial \hbar} \int D\phi \ \mathcal{O}_{i_1} \ldots \mathcal{O}_{i_n} \exp\left(\frac{i}{\hbar} \left\{ Q, \int V \right\} \right) = 0 \ . \tag{D.6}$$

Interestingly, from any scalar physical operator $\mathcal{O}^{(0)}$, i.e. from one that does not transform under coordinate transformation of M, where M is the manifold on which the theory is formulated, one can construct a series of non-local physical operators, which transform like forms. Integrating (D.3) over a space-like hypersurface gives

$$P_\alpha = \{Q, G_\alpha\} \ . \tag{D.7}$$

Consider

$$\mathcal{O}^{(1)}_\alpha := i\{G_\alpha, \mathcal{O}^{(0)}\} \ , \tag{D.8}$$

and calculate

$$\begin{aligned} \frac{d}{dx^\alpha} \mathcal{O}^{(0)} &= i[P_\alpha, \mathcal{O}^{(0)}] \\ &= i[\{Q, G_\alpha\}, \mathcal{O}^{(0)}] \\ &= \pm i\{\{G_\alpha, \mathcal{O}^{(0)}\}, Q\} - i\{\{\mathcal{O}^0, Q\}, G_\alpha\} \\ &= \{Q, \mathcal{O}^1\} \ . \end{aligned} \tag{D.9}$$

Let $\mathcal{O}^1 := \mathcal{O}^1_\alpha \mathrm{d}x^\alpha$, then

$$\mathrm{d}O^0 = \{Q, \mathcal{O}^1\} \ . \tag{D.10}$$

Integrating this equation over a closed curve γ in M gives

$$\left\{Q, \int_\gamma \mathcal{O}^{(1)}\right\} = 0 \ . \tag{D.11}$$

Therefore, the set of operators $\int_\gamma \mathcal{O}^{(1)}$ are (non-local) physical operators. Repeating this procedure gives

$$\begin{aligned}
\{Q, \mathcal{O}^{(0)}\} &= 0 \\
\{Q, \mathcal{O}^{(1)}\} &= \mathrm{d}\mathcal{O}^{(0)} \\
\{Q, \mathcal{O}^{(2)}\} &= \mathrm{d}\mathcal{O}^{(1)} \\
&\cdots \\
\{Q, \mathcal{O}^{(n)}\} &= \mathrm{d}\mathcal{O}^{(n-1)} \\
0 &= \mathrm{d}\mathcal{O}^{(n)} \ ,
\end{aligned} \tag{D.12}$$

where n is the dimension of M. Hence, the integral of $\mathcal{O}^{(p)}$ over a p-dimensional submanifold of M is physical. In particular, we have

$$\left\{Q, \int_M \mathcal{O}^{(n)}\right\} = 0 \ . \tag{D.13}$$

This implies that we can add terms $t^a\mathcal{O}_a^{(n)}$, with t^a arbitrary coupling constants to the Lagrangian, and the "deformed" theory then will still be cohomological.

D.2 $\mathcal{N}=(2,2)$ supersymmetry in 1+1 dimensions

The goal of this appendix is to construct the B-type topological string. This can be done by twisting an $\mathcal{N}=(2,2)$ supersymmetric theory in two (real) dimensions and then coupling the twisted theory to gravity. Let us therefore start by studying two-dimensional $\mathcal{N}=(2,2)$ theories. Since we will only be interested in theories living on complex one dimensional manifolds, which locally look like $\mathbb{C}$, we will concentrate on field theories on $\mathbb{C}$ with coordinates $z = x^1 + ix^0$. Here ix^0 can be understood as Euclidean time.

The Lorentz group on $\mathbb{C}$ is given by $U(1)$, and Weyl spinors have only one complex component[1]. On the other hand, these spinors transform under the $U(1)$ Lorentz group and one can classify them according to whether they have positive or negative $U(1)$-charge. A theory with p spinor supercharges of positive and q spinor supercharges of negative charge is said to be a $\mathcal{N}=(p,q)$ supersymmetric field theory.

Here we study $\mathcal{N}=(2,2)$ theories which are best formulated on *superspace* with coordinates $z, \bar{z}, \theta^\pm, \bar{\theta}^\pm$, where $\theta^\pm$ are Grassmann variables satisfying $(\theta^\pm)^* = \bar{\theta}^\mp$. The superscript $\pm$ indicates how the spinors transform under Lorentz transformation, $z \to z' = e^{i\alpha}z$, namely

$$\theta^\pm \to (\theta^\pm)' = e^{\pm i\alpha/2}\theta^\pm \quad , \quad \bar{\theta}^\pm \to (\bar{\theta}^\pm)' = e^{\pm i\alpha/2}\bar{\theta}^\pm \ . \tag{D.14}$$

[1] For a detailed description of spinors in various dimensions see appendix A.2.

Functions that live on superspace are called *superfields* and because of the Grassmannian nature of the fermionic coordinates they can be expanded as

$$\Psi(z,\bar{z},\theta^+,\theta^-,\bar{\theta}^+,\bar{\theta}_-)=\phi(z,\bar{z})+\theta^+\psi_+(z,\bar{z})+\theta^-\psi_-(z,\bar{z})+\theta^+\theta^-F(z,\bar{z})+\dots \quad \text{(D.15)}$$

Symmetries and the algebra

Having established superspace, consisting of bosonic coordinates $z,\bar{z}$ and fermionic coordinates $\theta^\pm,\bar{\theta}^\pm$ we might ask for the symmetry group of this space. In other words we are interested in the linear symmetries which leave the measure $\mathrm{d}z\mathrm{d}\bar{z}\mathrm{d}\theta^+\mathrm{d}\theta^-\mathrm{d}\bar{\theta}^+\mathrm{d}\bar{\theta}^-$ invariant. Clearly, part of this symmetry group is the two-dimensional Poincaré symmetry. The translations are generated by

$$\begin{aligned} H &= -i\frac{d}{d(ix^0)} = -i(\partial_z-\partial_{\bar{z}})\ , \\ P &= -i\frac{d}{dx^1} = -i(\partial_z+\partial_{\bar{z}})\ . \end{aligned}$$

We saw already how the spinors $\theta^\pm,\bar{\theta}^\pm$ transform under Lorentz transformation, so the generator reads

$$M = 2z\partial_z - 2\bar{z}\partial_{\bar{z}} + \theta^+\frac{d}{d\theta^+} - \theta^-\frac{d}{d\theta^-} + \bar{\theta}^+\frac{d}{d\bar{\theta}^+} - \bar{\theta}^-\frac{d}{d\bar{\theta}^-}\ , \quad \text{(D.16)}$$

where M is normalised in such a way that $e^{2\pi iM}$ rotates the Grassmann variables once and the complex variables twice. Other linear transformations are the translation of the fermionic coordinates $\theta^\pm$ generated by $\frac{\partial}{\partial\theta^\pm}$ and a change of bosonic coordinates $z\to z'=z+\epsilon\theta$, generated by the eight operators $\theta^\pm\partial_z,\theta^\pm\partial_{\bar{z}},\bar{\theta}^\pm\partial_z,\bar{\theta}^\pm\partial_{\bar{z}}$. From these various symmetry generators one defines differential operators on superspace,

$$\mathcal{Q}_\pm := \frac{\partial}{\partial\theta^\pm} + i\bar{\theta}^\pm\partial_\pm\ , \quad \text{(D.17)}$$

$$\overline{\mathcal{Q}}_\pm := -\frac{\partial}{\partial\bar{\theta}^\pm} - i\theta^\pm\partial_\pm\ , \quad \text{(D.18)}$$

$$D_\pm := \frac{\partial}{\partial\theta^\pm} - i\bar{\theta}^\pm\partial_\pm\ , \quad \text{(D.19)}$$

$$\overline{D}_\pm := -\frac{\partial}{\partial\bar{\theta}^\pm} + i\theta^\pm\partial_\pm\ , \quad \text{(D.20)}$$

where we denoted $\partial_+ := \partial_z$ and $\partial_- := \partial_{\bar{z}}$.

There are two more interesting linear symmetry transformations, known as the vector and axial R-rotations of a superfield. They are defined as

$$\begin{aligned} R_V(\alpha):\quad & (\theta^+,\bar{\theta}^+)\to(e^{-i\alpha}\theta^+,e^{i\alpha}\bar{\theta}^+)\quad,\quad(\theta^-,\bar{\theta}^-)\to(e^{-i\alpha}\theta^-,e^{i\alpha}\bar{\theta}^-)\ , \\ R_A(\alpha):\quad & (\theta^+,\bar{\theta}^+)\to(e^{-i\alpha}\theta^+,e^{i\alpha}\bar{\theta}^+)\quad,\quad(\theta^-,\bar{\theta}^-)\to(e^{i\alpha}\theta^-,e^{-i\alpha}\bar{\theta}^-)\ , \end{aligned}$$

with the corresponding operators

$$\begin{aligned} F_V &= -\theta^+ \frac{d}{d\theta^+} - \theta^- \frac{d}{d\theta^-} + \bar{\theta}^+ \frac{d}{d\bar{\theta}^+} + \bar{\theta}^- \frac{d}{d\theta^-} \ , \\ F_A &= -\theta^+ \frac{d}{d\theta^+} + \theta^- \frac{d}{d\theta^-} + \bar{\theta}^+ \frac{d}{d\bar{\theta}^+} - \bar{\theta}^- \frac{d}{d\theta^-} \ . \end{aligned} \tag{D.21}$$

Of course, a superfield Ψ might also transform under these transformations,

$$e^{i\alpha F_V} : \Psi(x^\mu, \theta^\pm, \bar{\theta}^\pm) \ \mapsto \ e^{i\alpha q_V} \Psi(x^\mu, e^{-i\alpha}\theta^\pm, e^{i\alpha}\bar{\theta}^\pm) \ , \tag{D.22}$$

$$e^{i\beta F_A} : \Psi(x^\mu, \theta^\pm, \bar{\theta}^\pm) \ \mapsto \ e^{i\beta q_A} \Psi(x^\mu, e^{\mp i\beta}\theta^\pm, e^{\pm i\beta}\bar{\theta}^\pm) \ , \tag{D.23}$$

where q_V and q_A are known as the *vector and axial R-charge* of Ψ.

From the operators constructed so far it is easy to derive the commutation relations. One obtains the algebra of an $\mathcal{N} = (2,2)$ supersymmetric theory

$$\begin{aligned} [M, H] &= -2P \quad , \quad [M, P] = -2H \\ \mathcal{Q}_+^2 = \mathcal{Q}_-^2 &= \overline{\mathcal{Q}}_+^2 = \overline{\mathcal{Q}}_-^2 = 0 \\ \{\mathcal{Q}_\pm, \overline{\mathcal{Q}}_\pm\} &= H \pm P \\ [M, \mathcal{Q}_\pm] &= \mp Q_\pm \quad , \quad [M, \overline{\mathcal{Q}}_\pm] = \mp \overline{\mathcal{Q}}_\pm \\ [F_V, \mathcal{Q}_\pm] &= -\mathcal{Q}_\pm \quad , \quad [F_V, \overline{\mathcal{Q}}_\pm] = +\overline{\mathcal{Q}}_\pm \\ [F_A, \mathcal{Q}_\pm] &= \mp \mathcal{Q}_\pm \quad , \quad [F_A, \overline{\mathcal{Q}}_\pm] = \pm \overline{\mathcal{Q}}_\pm \ . \end{aligned} \tag{D.24}$$

Chiral superfields

A *chiral superfield* is a function on superspace that satisfies

$$\overline{D}_\pm \Phi = 0 \ , \tag{D.25}$$

and fields satisfying

$$D_\pm \Upsilon = 0 \tag{D.26}$$

are called *anti-chiral superfields*. Note that the complex conjugate of a chiral superfield is anti-chiral and vice-versa. A chiral superfield has the expansion

$$\Phi(z, \theta^\pm, \bar{\theta}_\pm) = \phi(w, \bar{w}) + \theta^+ \psi_+(w, \bar{w}) + \theta^- \psi_-(w, \bar{w}) + \theta^+ \theta^- F(w, \bar{w}) \ , \tag{D.27}$$

where $w := z - i\theta^+\bar{\theta}^+$ and $\bar{w} := \bar{z} - i\theta^-\bar{\theta}^-$. Note that a $\mathcal{Q}$-transformed chiral field is still chiral, because the $\mathcal{Q}$-operators and the D-operators anti-commute.

Supersymmetric actions

We are interested in actions that are invariant under the supersymmetry transformation

$$\delta = \epsilon^+ \mathcal{Q}_+ + \epsilon^- \mathcal{Q}_- + \bar{\epsilon}^+ \overline{\mathcal{Q}}_+ + \bar{\epsilon}^- \overline{\mathcal{Q}}_- \ . \tag{D.28}$$

Let $K(\Psi_i, \bar{\Psi}_i)$ be a real differentiable function of superfields Ψ_i and consider the quantity

$$\int \mathrm{d}^2 z \ \mathrm{d}^4\theta \ K(\Psi_i, \bar{\Psi}_i)) := \int \mathrm{d}z \ \mathrm{d}\bar{z} \ \mathrm{d}\theta^+ \mathrm{d}\theta^- \mathrm{d}\bar{\theta}^+ \mathrm{d}\bar{\theta}^- K(\Psi_i, \bar{\Psi}_i) \ . \tag{D.29}$$

Functionals of this form are called *D-terms* and it is not hard to see that they are invariant under the supersymmetry transformation (D.28). If we require that for any Ψ_i with charges q_V^i, q_A^i the complex conjugate field $\bar{\Psi}_i$ has opposite charges $-q_V, -q_A$, the D-term is also invariant under the axial and vector R-symmetry.
Another invariant under supersymmetry can be constructed from chiral superfields Φ_i. Let $W(\Phi_i)$ be a *holomorphic* function of the Φ_i, called the superpotential, and consider

$$\int \mathrm{d}^2 z\ \mathrm{d}^2\theta\ W(\Phi_i) := \int \mathrm{d}^2 z\ \mathrm{d}\theta^+ \mathrm{d}\theta^-\ W(\Phi_i)\Big|_{\bar{\theta}^\pm=0} \tag{D.30}$$

This invariant is called an *F-term*. Clearly, this term is invariant under the axial R-symmetry if the Φ_i have $q_A = 0$. The vector R-symmetry, however, is only conserved if we can assign vector R-charge two to the superpotential $W(\Phi)$. For monomial superpotentials this is always possible.

It is quite interesting to analyse the action (D.29) in the case in which K is a function of chiral superfields Φ_i and their conjugates. As to do so we want to spell out the action in terms of the component fields and keep only those fields that contain the fields ϕ_i. The coefficient of $\theta^+\theta^-\bar{\theta}^+\bar{\theta}^-$ can be read off to be

$$\frac{dK}{d\phi^i}\partial_+\partial_-\phi^i + \frac{d^2K}{d\phi^i d\phi^j}\partial_+\phi^i\partial_-\phi^j = -\frac{d^2K}{d\phi^i d\bar{\phi}^j}\partial_+\bar{\phi}^j\partial_-\phi^i + d\left(\frac{dK}{d\phi^i}\partial_-\phi^i\right) . \tag{D.31}$$

Of course, the last term vanishes under the integral. The $\bar{\phi}^i$-dependent terms give the same expression with $+$ and $-$ interchanged. This implies that the $\phi, \bar{\phi}$-dependent part of the D-term action can be written as

$$S_\phi = -\int \mathrm{d}^2 z\ g_{i\bar{j}}\eta^{\alpha\beta}\partial_\alpha\phi^i\partial_\beta\bar{\phi}^j , \tag{D.32}$$

with $\eta^{+-} = \eta^{-+} = 2$, $\eta^{++} = \eta^{--} = 0$ and

$$g_{i\bar{j}}(\phi, \bar{\phi}) = \frac{d^2K}{d\phi^i d\bar{\phi}^j} . \tag{D.33}$$

If we interpret the ϕ^i as coordinates on some target space $\mathcal{M}$, we find that this space carries as metric $g_{i\bar{j}}$ which is Kähler with Kähler potential K. Of course, one could also write down all the other terms appearing in the action (D.29), but the expression would be rather lengthy. One the other hand it does not contain derivatives of the F_i, and all F_i appear at most quadratically. Therefore, we can integrate over F_i in the path integral and the result will be to substitute the value it has according to its equation of motion. Then the action turns into the rather simple form

$$L = -g_{i\bar{j}}\partial^\alpha\phi^i\partial_\alpha\bar{\phi}^j - 2ig_{i\bar{j}}\bar{\psi}_-^j\Delta_+\psi_-^i - 2ig_{i\bar{j}}\bar{\psi}_+^j\Delta_-\psi_+^i - R_{i\bar{j}k\bar{l}}\psi_+^i\psi_-^k\bar{\psi}_+^j\bar{\psi}_-^l , \tag{D.34}$$

where

$$\begin{aligned} R_{i\bar{j}k\bar{l}} &= g^{m\bar{n}}\partial_{\bar{l}}g_{m\bar{j}}\partial_k g_{\bar{n}i} - \partial_k\partial_{\bar{l}}g_{i\bar{j}} , \\ \Delta_\pm\psi^i &= \partial_\pm\psi^i + \Gamma^i_{jk}\partial_\pm\phi^j\psi^k , \\ \Gamma^i_{jk} &= g^{i\bar{l}}\partial_k g_{\bar{l}j} . \end{aligned} \tag{D.35}$$

Quantum theory and anomalies

When studying a quantum field theory it is always an interesting question whether the symmetries of the classical actions persist on the quantum level, or whether they are anomalous. Here we want to check whether the quantum theory with action (D.29), and with chiral superfields Φ_i instead of the general fields Ψ_i, is still invariant under the vector and axial symmetries. One possibility to analyse the anomalies of a quantum theory is to study the path integral measure,

$$\prod_i D\phi^i D\psi_+^i D\psi_-^i DF^i \times c.c. \; . \tag{D.36}$$

If we take the charges q_V, q_A of all the Φ_i to zero, the ϕ^i do not transform under R-symmetry and, as we just saw, the F^i can be integrated out, so it remains to check whether the fermion measure is invariant. To proceed one first of all assumes that the size of the target manifold is large compared to the generic size of the world-sheet. In this case the Riemann curvature will be small and one can neglect the last term in the action (D.34). Then the path integral over ψ_- has the form

$$\int D\psi_- D\bar{\psi}_- \exp(\bar{\psi}_-, \Delta_+\psi_-) \; , \tag{D.37}$$

where the inner product is defined as $(a,b) := \int_{\mathbb{C}} a^i b_i$. A quantity that will be important in what follows is the index k of the operator Δ_+,

$$k = \dim \mathrm{Ker}\Delta_+ - \dim \mathrm{Ker}\Delta_+^\dagger \; . \tag{D.38}$$

This quantity is actually a topological invariant, and the Atiyah-Singer index theorem tells us that in this simple case it can be written as

$$k = \int_{\phi(\Sigma)} c_1(\mathcal{M}) \; . \tag{D.39}$$

Here Σ is the world-sheet, i.e. it is $\mathbb{C}$ in the case we studied so far, and $\mathcal{M}$ denotes the target space. We will not address the problem of how to study the fermion measure in detail, but we only list the results. First of all, because of the Grassmannian nature of the integrals over the fermionic variables, one can show that the quantity $\int D\psi_- D\bar{\psi}_- \exp(\bar{\psi}_-, \Delta_+\psi_-)$ does actually vanish, unless $k = 0$. In order to obtain a non-zero result for finite k one has to insert fermions into the path integral,

$$\int D\psi_+ D\psi_- D\bar{\psi}_+ D\bar{\psi}_- \left(g_{i_1\bar{j}_1}\psi_-^{i_1}(z_1)\bar{\psi}_+^{j_1}(z_1)\dots g_{i_k\bar{j}_k}\psi_-^{i_k}(z_k)\bar{\psi}_+^{j_k}(z_k)\right) e^{-S} \; . \tag{D.40}$$

It is easy to see that these quantities are invariant under the vector symmetry, but the axial symmetry is broken unless $k = 0$. This is a first indication why Calabi-Yau manifolds are particularly useful for the topological string, since $k = 0$ implies that $c_1(\mathcal{M}) = 0$.

D.3 The topological B-model

The starting point of this section is once again the action

$$S = \int_\Sigma \mathrm{d}^2 z \mathrm{d}^4\theta K(\Phi^i, \bar{\Phi}^i) \tag{D.41}$$

with chiral superfields Φ^i, now formulated on an arbitrary Riemann surface Σ.

The $\mathcal{N} = (2,2)$ supersymmetric field theories constructed so far are not yet cohomological theories. Note however, that

$$\begin{aligned} \{\overline{\mathcal{Q}}_+ + \overline{\mathcal{Q}}_-, \mathcal{Q}_+ - \mathcal{Q}_-\} &= 2H \ , \\ \{\overline{\mathcal{Q}}_+ + \overline{\mathcal{Q}}_-, \mathcal{Q}_+ + \mathcal{Q}_-\} &= 2P \ , \end{aligned}$$

so if we define

$$Q_B := \overline{\mathcal{Q}}_+ + \overline{\mathcal{Q}}_- \tag{D.42}$$

then

$$Q_B^2 = 0 \ , \tag{D.43}$$

and H and P are Q_B-exact. However, one of the central properties of a cohomological theory is the fact that it is independent of the world-sheet metric, i.e. one should be able to define it for an arbitrary world-sheet metric. This can be done on the level of the Lagrangian, by just replacing partial derivatives with respect to the world-sheet coordinates by covariant derivatives. However, one runs into trouble if one wants to maintain the symmetry of the $\mathcal{N} = (2,2)$ theory. In particular, the supersymmetries should be global symmetries, not local ones. This amounts to saying that in the transformation $\delta\Phi^i = \epsilon^+ \mathcal{Q}_+ \Phi^i$ the spinor ϵ^+ has to be covariantly constant with respect to the world-sheet metric. But for a general metric on the world-sheet there are no covariantly constant spinors. So, it seems to be impossible to construct a cohomological theory from the $\mathcal{N} = (2,2)$ supersymmetric one.

On the other hand, for a symmetry generated by a bosonic generator the infinitesimal parameter is simply a number. In other words, it lives in the trivial bundle $\Sigma \times \mathbb{C}$ and can be chosen to be constant. This gives us a hint on how the above problem can be solved. If we can somehow arrange for some of the $\mathcal{Q}$-operators to live in a trivial bundle, the corresponding symmetry can be maintained. Clearly, the type of bundle in which an object lives is defined by its charge under the Lorentz symmetry. We are therefore led to the requirement to modify the Lorentz group in such a way that the $\mathcal{Q}_B$-operator lives in a trivial bundle, i.e. has spin zero. This can actually be achieved by defining

$$M_B := M - F_A \tag{D.44}$$

to be the generator of the Lorentz group. One then finds the following commutation relations

$$\begin{aligned} [M_B, \mathcal{Q}_+] = -2\mathcal{Q}_+ \quad &, \quad [M_B, \mathcal{Q}_-] = 2\mathcal{Q}_- \\ [M_B, \overline{\mathcal{Q}}_+] = 0 \quad &, \quad [M_B, \overline{\mathcal{Q}}_-] = 0 \ . \end{aligned}$$

Note that now the operator Q_B indeed is a scalar, and therefore the corresponding symmetry can be defined on an arbitrary curved world-sheet. This construction is called *twisting* and we arrive at the conclusion that the twisted theory truly is a cohomological field theory. Of course, on the level of the Lagrangian one has to replace partial derivatives by covariant ones for general metrics on the world-sheet, but now the covariant derivatives have to be covariant with respect to the modified Lorentz group.

So far our discussion was on the level of the algebra and rather general. Let us now come back to the action (D.41) analysed in the last section. Note first of all that the B-type twist (D.44) involves the axial vector symmetry, which remains valid on the quantum level only if we take the target space to have $c_1 = 0$. Therefore, we will take the target space $\mathcal{M}$ to be a Calabi-Yau manifold from now on.

Let us then study in which bundles the various fields of our theory live after twisting. It is not hard to see that

$$\begin{aligned}
\psi_+^i &\in \Lambda^{1,0}(\Sigma) \otimes \phi^*(T^{(1,0)}(\mathcal{M})) \ , \\
\psi_-^i &\in \Lambda^{0,1}(\Sigma) \otimes \phi^*(T^{(1,0)}(\mathcal{M})) \ , \\
\bar{\psi}_+^i &\in \phi^*(T^{(0,1)}(\mathcal{M})) \ , \\
\bar{\psi}_-^i &\in \phi^*(T^{(0,1)}(\mathcal{M})) \ ,
\end{aligned}$$

where $\in$ means "is a section of". This simply says that e.g. ψ_+^i transforms as a $(1,0)$-form on the world-sheet and as a holomorphic vector in space time, whereas e.g. $\bar{\psi}_+^i$ transforms as a scalar on the world-sheet, but as an anti-holomorphic vector in space-time. It turns out that the following reformulation is convenient,

$$\begin{aligned}
\eta^{\bar{i}} &:= \bar{\psi}_+^i + \bar{\psi}_-^i \ , \\
\theta_i &:= g_{i\bar{j}}(\bar{\psi}_+^j - \bar{\psi}_-^j) \ , \\
\rho_z^i &:= \psi_+^i \ , \\
\rho_{\bar{z}}^i &:= \psi_-^i \ .
\end{aligned}$$

The twisted $\mathcal{N} = (2,2)$ supersymmetric theory with a twist (D.44) and action (D.41) is called the *B-model*. Its Lagrangian can be rewritten in terms of the new fields,

$$L = -t\left(g_{i\bar{j}}\eta^{\alpha\beta}\partial_\alpha\phi^i\partial_\beta\bar{\phi}^j + i g_{i\bar{j}}\eta^{\bar{j}}(\Delta_{\bar{z}}\rho_z^i + \Delta_z\rho_{\bar{z}}^i) + i\theta_i(\Delta_{\bar{z}}\rho_z^i - \Delta_z\rho_{\bar{z}}^i) + \frac{1}{2}R_{i\bar{j}k}{}^{l}\rho_z^i\rho_{\bar{z}}^k\eta^{\bar{j}}\theta_l\right) \ , \tag{D.45}$$

where t is some coupling constant. Here we still used a flat metric on Σ to write the Lagrangian, but from our discussion above we know that we can covariantise it using an arbitrary metric on the world-sheet, without destroying the symmetry generated by Q_B. The Lagrangian can be rewritten in the form

$$L = -it\{Q_B, V\} - t\left(i\theta_i(\Delta_{\bar{z}}\rho_z^i - \Delta_z\rho_{\bar{z}}^i) + \frac{1}{2}R_{i\bar{j}k}{}^{l}\rho_z^i\rho_{\bar{z}}^k\eta^{\bar{j}}\theta_l\right) \ , \tag{D.46}$$

with

$$V = g_{i\bar{j}} \left(\rho^i_z \partial_{\bar{z}} \bar{\phi}^j + \rho^i_{\bar{z}} \partial_z \bar{\phi}^j \right) \ . \tag{D.47}$$

Now it seems as if the B-model was not cohomological after all, since the second term of (D.45) is not Q_B-exact. However, it is anti-symmetric in z and $\bar{z}$, and therefore it can be understood as a differential form. The integral of such a form is independent of the metric and therefore the B-model is a topological quantum field theory.

We mention without proof that the B-model does depend on the complex structure of the target space, but it is actually independent of its Kähler structure [146]. The idea of the proof is that the variation of the action with respect to the Kähler form is Q_B-exact.

Furthermore, the t-dependence of the second term in (D.45) can be eliminated by a rescaling of θ_i. If one studies only correlation function which are homogeneous in θ, the path integral only changes by an overall factor of t to some power. Apart from this prefactor the correlation function is independent of t, as can be seen by performing a calculation similar to the one in (D.6). But this means that to calculate these correlation functions one can take the limit in which t is large and the result will be exact. This fact has interesting consequences. For example consider the equations of motion for ϕ and $\bar{\phi}$,

$$\partial_z \phi^i = \partial_{\bar{z}} \phi^i = \partial_z \bar{\phi}^i = \partial_{\bar{z}} \bar{\phi}^i = 0 \ , \tag{D.48}$$

which only have constant maps as their solutions. Since the "classical limit" $t \to \infty$ gives the correct result for any t, up to an overall power of t, we find that in the path integral for ϕ we only have to integrate over the space of constant maps, which is simply the target space $\mathcal{M}$ itself.

A set of metric independent local operators can be constructed from

$$V := V^{j_1 \ldots j_q}_{\bar{i}_1 \ldots \bar{i}_p}(\phi, \bar{\phi}) \mathrm{d}\bar{\phi}^{\bar{i}_1} \wedge \ldots \mathrm{d}\bar{\phi}^{\bar{i}_p} \frac{\partial}{\partial \phi^j_1} \cdots \frac{\partial}{\partial \phi^j_q} \tag{D.49}$$

as

$$\mathcal{O}_V := V^{j_1 \ldots j_q}_{\bar{i}_1 \ldots \bar{i}_p}(\phi, \bar{\phi}) \eta^{\bar{i}_1} \ldots \eta^{\bar{i}_p} \theta_{j_1} \ldots \theta_{j_q} \ . \tag{D.50}$$

The transformation laws

$$\begin{aligned} \{Q_B, \phi^i\} &= 0 \quad , \quad \{Q_B, \bar{\phi}^i\} = -\eta^{\bar{i}} \ , \\ \{Q_B, \theta_i\} &= 0 \quad , \quad \{Q_B, \eta^{\bar{i}}\} = 0 \ , \end{aligned}$$

can then be used to show that

$$\{Q_B, \mathcal{O}_V\} = -\mathcal{O}_{\bar{\partial} V} \ . \tag{D.51}$$

We see that Q_B can be understood as the Dolbeault exterior derivative $\bar{\partial}$ and the physical operators are in one-to-one correspondence with the Dolbeault cohomology classes.

D.4 The B-type topological string

So far the metric on the world-sheet was taken to be a fixed background metric. To transform the B-model to a topological string theory one has actually to integrate over all possible metrics on the world-sheet. If one wants to couple an ordinary field theory to gravity the following steps have to be performed.

- The Lagrangian has to be rewritten in a covariant way, by replacing the flat metrics by dynamical ones, introducing covariant derivatives and multiplying the measure by $\sqrt{\det h}$
- One has to add an Einstein-Hilbert term to the action, plus possibly other terms, to maintain the original symmetries of the theory.
- The path integral measure has to include a factor Dh, integrating over all possible metrics.

Here we only provide a sketch of how the last step of this procedure might be performed. We start by noting that, once we include the metric in the Lagrangian, the theory becomes conformal. But this means that one can use the methods of ordinary string theory to calculate the integral over all conformally equivalent metrics. An important issue that occurs at this point in standard string theory is the conformal anomaly. To understand this in our context let us review the twisting from a different perspective. We start from the energy momentum tensor $T_{\alpha\beta}$, which in conformal theories is known to have the structure $T_{z\bar{z}} = T_{\bar{z}z} = 0$, $T_{zz} = T(z)$ and $T_{\bar{z}\bar{z}} = \bar{T}(\bar{z})$. One expands $T(z) = \sum_{m=-\infty}^{\infty} L_m z^{-m-2}$, and the Virasoro generators satisfy

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n} \ . \tag{D.52}$$

c is the central charge and it depends on the theory. Technical problems occur for non-zero c, since the equation of motion for the metric reads

$$\frac{\delta S}{\delta h_{\alpha\beta}} = T_{\alpha\beta} = 0 \ . \tag{D.53}$$

In conformal theories this equation is imposed as a constraint, i.e. one requires that a physical state $|\,\psi\rangle$ satisfies

$$L_m\,|\,\psi\rangle = 0 \quad \forall m \in \mathbb{Z} \ . \tag{D.54}$$

This is compatible with the Virasoro algebra only if $c = 0$. If $c \neq 0$ one speaks of a conformal anomaly. Let us then check whether we have a central charge in the case of the twisted theory.

Note from (D.21) that $F_V + F_A$ acts on objects with a + index, i.e. on left-moving quantities, whereas $F_V - F_A$ acts on objects with a minus index, i.e. on right-moving quantities. Therefore we define $F_L := F_V + F_A$ and $F_R := F_V - F_R$. It can be shown

that these two symmetries can be identified with the two components of a single global $U(1)$ current. To be more precise we have

$$F_L = \int_{z=0} J(z)\mathrm{d}z \ , \tag{D.55}$$

and similarly for F_R. Expanding the current as

$$J(z) = \sum_{m=-\infty}^{\infty} J_m z^{-m-1} \tag{D.56}$$

gives $F_L = 2\pi i J_0$. Now recall that $M_B = M - F_A = M - \frac{1}{2}(F_L - F_R)$ and from $M = 2\pi i(L_0 - \bar{L}_0)$ we find

$$L_{0,B} = L_0 - \frac{1}{2}J_0 \quad , \quad \bar{L}_{0,B} = \bar{L}_0 - \frac{1}{2}\bar{J}_0 \ . \tag{D.57}$$

This twisted Virasoro generator can be obtained from

$$T_B(z) = T(z) + \frac{1}{2}\partial J(z) \ , \tag{D.58}$$

with generators

$$L_{m,B} = L_m - \frac{1}{2}(m+1)J_m \tag{D.59}$$

The algebra of these modified generators can be calculated explicitly and one finds

$$[L_{m,B}, L_{n,B}] = (m-n)L_{m+n,B} \ . \tag{D.60}$$

We find that the central charge is automatically zero, and as a consequence the topological string is actually well-defined in any number of space-time dimensions.

Now we can proceed as in standard string theory, in other words we sum over all genera $\hat{g}$ of the world-sheet, integrate over the moduli space of a Riemann surface of genus $\hat{g}$ and integrate over all conformally equivalent metrics on the surface. In close analogy to what is done in the bosonic string the free energy at genus $\hat{g}$ of the B-model topological string is given by (c.f. Eq. (5.4.19) of [117])

$$F_{\hat{g}} := \int_{\mathcal{M}_{\hat{g}}} \left\langle \prod_{i=1}^{3\hat{g}-3} \left(\mathrm{d}m^i \wedge \mathrm{d}\bar{m}^{\bar{i}} \int_\Sigma G_{zz}(\mu_i)^z_{\bar{z}} \int_\Sigma G_{\bar{z}\bar{z}}(\bar{\mu}_{\bar{i}})^{\bar{z}}_z \right) \right\rangle \ , \tag{D.61}$$

where $\mathcal{M}_{\hat{g}}$ is the moduli space of a Riemann surface of genus $\hat{g}$. As usual the $(\mu_i)^z_{\bar{z}}$ are defined from the change of complex structure, $\mathrm{d}z^i \to \mathrm{d}z^i + \epsilon(\mu^i)^z_{\bar{z}}\mathrm{d}\bar{z}$, and the $\mathrm{d}m^i$ are the dual forms of the μ_i. Furthermore, the quantity G_{zz} is the Q-partner of the energy momentum tensor component T_{zz}. Interestingly, one can show that the $F_{\hat{g}}$ vanish for every $\hat{g} > 1$, unless the target space of the topological string is of dimension three.

This elementary definition of the B-type topological string is now the starting point for a large number of interesting applications. However, we will have to refrain from explaining further details and refer the interested reader to the literature [81], [133], [111].

Appendix E

Anomalies

Anomalies have played a fascinating role both in quantum field theory and in string theory. Many of the results described in the main text are obtained by carefully arranging a given theory to be free of anomalies. Here we provide some background material on anomalies and fix the notation. A more detailed discussion can be found in [P4]. General references are [11], [12], [13], [134] and [139]. In this appendix we work in Euclidean space.

E.1 Elementary features of anomalies

In order to construct a quantum field theory one usually starts from a classical theory, which is quantized by following one of several possible quantization schemes. Therefore, a detailed analysis of the classical theory is a crucial prerequisite for understanding the dynamics of the quantum theory. In particular, the symmetries and the related conservation laws should be mirrored on the quantum level. However, it turns out that this is not always the case. If the classical theory possesses a symmetry that cannot be maintained on the quantum level we speak of an *anomaly*.

A quantum theory containing a massless gauge field A is only consistent if it is invariant under the infinitesimal local gauge transformation

$$A'(x) = A(x) + D\epsilon(x) \ . \tag{E.1}$$

The invariance of the action can be written as

$$D_M(x)\frac{\delta S[A]}{\delta A_{aM}(x)} = 0 \ , \tag{E.2}$$

where $A = A_a T_a = A_{aM} T_a dx^M$. Then we can define a current corresponding to this symmetry,

$$J_a^M(x) := \frac{\delta S[A]}{\delta A_{aM}(x)} \ , \tag{E.3}$$

and gauge invariance (E.2) of the action tells us that this current is conserved,

$$D_M J_a^M(x) = 0 \ . \tag{E.4}$$

Suppose we consider a theory containing massless fermions ψ in the presence of an external gauge field A. In such a case the expectation value of an operator is defined as[1]

$$\langle \mathcal{O} \rangle = \frac{\int D\psi D\bar{\psi} \ \mathcal{O} \exp(-S[\psi, A])}{\int D\psi D\bar{\psi} \ \exp(-S[\psi, A])} \ , \tag{E.5}$$

and we define the quantity

$$\exp(-X[A]) := \int D\psi D\bar{\psi} \ \exp(-S[\psi, A]) \ . \tag{E.6}$$

Then it is easy to see that

$$\langle J_a^M(x) \rangle = \frac{\delta X[A]}{\delta A_{aM}(x)} \ . \tag{E.7}$$

An anomaly occurs if a symmetry is broken on the quantum level, or in other words if $X[A]$ is not gauge invariant, even though $S[\psi, A]$ is. The non-invariance of $X[A]$ can then be understood as coming from a non-trivial transformation of the measure. Indeed, if we have

$$D\psi D\bar{\psi} \to \exp\left(i \int d^d x \ \epsilon_a(x) G_a[x; A] \right) D\psi D\bar{\psi}, \tag{E.8}$$

then the variation of the functional (E.6) gives

$$\exp(-X[A]) \int d^d x \ D_M \langle J_a^M(x) \rangle \epsilon_a(x) = \int d^d x \int D\psi D\bar{\psi} [i G_a[x; A] \epsilon_a(x)] \exp(-S) \ . \tag{E.9}$$

This means that the quantum current will no longer be conserved, but we get a generalised version of (E.4),

$$D_M \langle J_a^M(x) \rangle = i G_a[x; A] \ . \tag{E.10}$$

$G_a[x; A]$ is called the *anomaly*.

Not every symmetry of an action has to be a local gauge symmetry. Sometimes there are global symmetries of the fields

$$\Phi' = \Phi + i\epsilon \Delta\Phi \ . \tag{E.11}$$

These symmetries lead to a conserved current as follows. As the action is invariant under (E.11), for

$$\Phi' = \Phi + i\epsilon(x) \Delta\Phi \tag{E.12}$$

[1] We work in Euclidean space after having performed a Wick rotation. Our conventions in the Euclidean are as follows: $S_M = iS_E$, $ix_M^0 = x_E^1$, $x_M^1 = x_E^2, \ldots x_M^{d-1} = x_E^d$; $i\Gamma_M^0 = \Gamma_E^1$, $\Gamma_M^1 = \Gamma_E^2, \ldots \Gamma_M^{d-1} = \Gamma_E^d$; $\Gamma_E := i^{\frac{d}{2}} \Gamma_E^1, \ldots, \Gamma_E^d$. For details on conventions in Euclidean space see [P3].

we get a transformation of the form

$$\delta S[\Phi] = -\int d^d x \; J^M(x)\partial_M \epsilon(x) \; . \tag{E.13}$$

If the fields Φ now are taken to satisfy the field equations then (E.13) has to vanish. Integrating by parts we find

$$\partial_M J^M(x) = 0 \; , \tag{E.14}$$

the current is conserved on shell.[2] Again this might no longer be true on the quantum level. An anomaly of a global symmetry is not very problematic. It simply states that the quantum theory is less symmetric than its classical origin. If on the other hand a local gauge symmetry is lost on the quantum level the theory is inconsistent. This comes about as the gauge symmetry of a theory containing massless spin-1 fields is necessary to cancel unphysical states. In the presence of an anomaly the quantum theory will no longer be unitary and hence useless. This gives a strong constraint for valid quantum theories as one has to make sure that all the local anomalies vanish.

The chiral anomaly

Consider the specific example of non-chiral fermions ψ in four dimensions coupled to external gauge fields $A = A_a T_a = A_{a\mu} T_a dx^\mu$ with Lagrangian

$$\mathcal{L} = \bar{\psi} i\gamma^\mu D_\mu \psi = \bar{\psi} i\gamma^\mu (\partial_\mu + A_\mu)\psi \; . \tag{E.15}$$

It is invariant under the global transformation

$$\psi' := \exp(i\epsilon\gamma_5)\psi \; , \tag{E.16}$$

with ϵ an arbitrary real parameter. This symmetry is called the *chiral symmetry*. The corresponding (classical) current is

$$J_5^\mu(x) = \bar{\psi}(x)\gamma^\mu\gamma_5\psi(x) \; ,$$

and it is conserved $\partial_\mu J_5^\mu = 0$, by means of the equations of motion. For this theory one can now explicitly study the transformation of the path integral measure [62], see [P4] for a review. The result is that

$$G[x; A] = \frac{1}{16\pi^2}\mathrm{tr}[\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}(x) F_{\rho\sigma}(x)] \; . \tag{E.17}$$

We conclude that the chiral symmetry is broken on the quantum level and we are left with what is known as the *chiral anomaly*

$$\partial_\mu \langle J_5^\mu(x)\rangle = \frac{i}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr} F_{\mu\nu}(x) F_{\rho\sigma}(x) \; . \tag{E.18}$$

[2]This can be generalized to theories in curved space-time, where we get $\nabla_M J^M(x) = 0$, with the Levi-Civita connection ∇.

The non-Abelian anomaly

Next we study a four-dimensional theory containing a Weyl spinor χ coupled to an external gauge field $A = A_a T_a$. Again we take the base manifold to be flat and four-dimensional. The Lagrangian of this theory is

$$\mathcal{L} = \bar{\chi} i \gamma^\mu D_\mu P_+ \chi = \bar{\chi} i \gamma^\mu (\partial_\mu + A_\mu) P_+ \chi \ . \tag{E.19}$$

It is invariant under the transformations

$$\begin{aligned} \chi' &= g^{-1}\chi \ , \\ A' &= g^{-1}(A+d)g \ , \end{aligned} \tag{E.20}$$

with the corresponding current

$$J_a^\mu(x) := i\bar{\chi}(x) T_a \gamma^\mu P_+ \chi(x) \ . \tag{E.21}$$

Again the current is conserved on the classical level, i.e. we have

$$D_\mu J_a^\mu(x) = 0 \ . \tag{E.22}$$

In order to check whether the symmetry is maintained on the quantum level, one once again has to study the transformation properties of the measure. The result of such a calculation (see for example [134], [109]) is[3]

$$D_\mu \langle J_a^\mu(x)\rangle = \frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}[T_a \partial_\mu (A_\nu \partial_\rho A_\sigma + \frac{1}{2} A_\nu A_\rho A_\sigma)] \ . \tag{E.23}$$

If the chiral fermions couple to Abelian gauge fields the anomaly simplifies to

$$D_\mu \langle J_a^\mu(x)\rangle = -\frac{i}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu^b \partial_\rho A_\sigma^c \cdot (q_a q_b q_c) = -\frac{i}{96\pi^2}\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^b F_{\rho\sigma}^c \cdot (q_a q_b q_c) \ . \tag{E.24}$$

Here we used $T_a = iq_a$ which leads to $D = d + iq_a A_a$, the correct covariant derivative for Abelian gauge fields. The index a now runs from one to the number of Abelian gauge fields present in the theory.

Consistency conditions and descent equations

In this section we study anomalies related to local gauge symmetries from a more abstract point of view. We saw above that a theory containing massless spin-1 particles has to be invariant under local gauge transformations to be a consistent quantum theory. These transformations read in their infinitesimal form $A_\mu(y) \to A_\mu(y) + D_\mu \epsilon(y)$. This can be rewritten as $A_{\mu b}(y) \to A_{\mu b}(y) - i\int d^4x \ \epsilon_a(x)\mathcal{T}_a(x) A_{\mu b}(y)$, with

$$-i\mathcal{T}_a(x) := -\frac{\partial}{\partial x^\mu}\frac{\delta}{\delta A_{\mu a}(x)} - C_{abc} A_{\mu b}(x)\frac{\delta}{\delta A_{\mu c}(x)} \ . \tag{E.25}$$

[3] Note that this anomaly is actually purely imaginary as it should be in Euclidean space, since it contains three factors of $T_a = -it_a$.

Using this operator we can rewrite the divergence of the quantum current (E.10) as

$$\mathcal{T}_a(x)X[A] = G_a[x;A] \; . \tag{E.26}$$

It is easy to show that the generators $\mathcal{T}_a(x)$ satisfy the commutation relations

$$[\mathcal{T}_a(x), \mathcal{T}_b(y)] = iC_{abc}\mathcal{T}_c(x)\delta(x-y) \; . \tag{E.27}$$

From (E.26) and (E.27) we derive the *Wess-Zumino consistency condition* [136]

$$\mathcal{T}_a(x)G_b[y;A] - \mathcal{T}_b(y)G_a[x;A] = iC_{abc}\delta(x-y)G_c[x;A] \; . \tag{E.28}$$

This condition can be conveniently reformulated using the BRST formalism. We introduce a ghost field $c(x) := c_a(x)T_a$ and define the BRST operator by

$$\begin{aligned} sA &:= -Dc \; , & \text{(E.29)} \\ sc &:= -\frac{1}{2}[c,c] \; . & \text{(E.30)} \end{aligned}$$

s is nilpotent, $s^2 = 0$, and satisfies the Leibnitz rule $s(AB) = s(A)B \pm As(B)$, where the minus sign occurs if A is a fermionic quantity. Furthermore, it anticommutes with the exterior derivative, $sd + ds = 0$. Next we define the anomaly functional

$$G[c;A] := \int d^4x \; c_a(x)G_a[x;A] \; . \tag{E.31}$$

For our example (E.23) we get

$$G[c;A] = -\frac{i}{24\pi^2}\int \mathrm{tr}\left\{c \; d\left[AdA + \frac{1}{2}A^3\right]\right\} \; . \tag{E.32}$$

Using the consistency condition (E.28) it is easy to show that

$$sG[c;A] = 0 \; . \tag{E.33}$$

Suppose $G[c;A] = sF[A]$ for some local functional $F[A]$. This certainly satisfies (E.33) since s is nilpotent. However, it is possible to show that all these terms can be cancelled by adding a local functional to the action. This implies that anomalies of quantum field theories are characterized by the cohomology groups of the BRST operator. They are the local functionals $G[c;A]$ of ghost number one satisfying the Wess-Zumino consistency condition (E.33), which cannot be expressed as the BRST operator acting on some local functional of ghost number zero.

Solutions to the consistency condition can be constructed using the Stora-Zumino descent equations. To explain this formalism we take the dimension of space-time to be $2n$. Consider the $(2n+2)$-form

$$\mathrm{ch}_{n+1}(A) := \frac{1}{(n+1)!}\mathrm{tr}\left(\frac{iF}{2\pi}\right)^{n+1} \; , \tag{E.34}$$

which is called the (n+1)-th *Chern character*[4]. As F satisfies the Bianchi identity we have $dF = [A, F]$, and therefore, $\mathrm{tr}F^{n+1}$ is closed, $d\ \mathrm{tr}F^{n+1} = 0$. One can show (see [P4] for details and references) that on any coordinate patch the Chern character can be written as

$$\mathrm{ch}_{n+1}(A) = d\Omega_{2n+1}, \tag{E.35}$$

with

$$\Omega_{2n+1}(A) = \frac{1}{n!}\left(\frac{i}{2\pi}\right)^{n+1}\int_0^1 dt\ \mathrm{tr}(AF_t^n)\ . \tag{E.36}$$

Here $F_t := dA_t + \frac{1}{2}[A_t, A_t]$, and $A_t := tA$ interpolates between 0 and A, if t runs from 0 to 1. $\Omega_{2n+1}(A)$ is known as the *Chern-Simons form* of $\mathrm{ch}_{n+1}(A)$. From the definition of the BRST operator and the gauge invariance of $\mathrm{tr}F^{n+1}$ we find that $s(\mathrm{tr}F^{n+1}) = 0$. Hence $d(s\Omega_{2n+1}(A)) = -sd\Omega_{2n+1}(A) = -s(\mathrm{ch}_{n+1}(A)) = 0$, and, from Poincaré's lemma,

$$s\Omega_{2n+1}(A) = d\Omega^1_{2n}(c, A)\ . \tag{E.37}$$

Similarly, $d(s\Omega^1_{2n}(c, A)) = -s^2\Omega_{2n+1}(A) = 0$, and therefore

$$s\Omega^1_{2n}(c, A) = d\Omega^2_{2n-1}(c, A)\ . \tag{E.38}$$

(E.37) and (E.38) are known as the *descent equations*. They imply that the integral of $\Omega^1_{2n}(c, A)$ over $2n$-dimensional space-time is BRST invariant,

$$s\int_{M_{2n}} \Omega^1_{2n}(c, A) = 0\ . \tag{E.39}$$

But this is a local functional of ghost number one, so it is identified (up to possible prefactors) with the anomaly $G[c; A]$. Thus, we found a solution of the Wess-Zumino consistency condition by integrating the two equations $d\Omega_{2n+1}(A) = \mathrm{ch}_{n+1}(A)$ and $d\Omega^1_{2n}(c, A) = s\Omega_{2n+1}(A)$. As an example let us consider the case of four dimensions. We get

$$\Omega_5(A) = \frac{1}{2}\left(\frac{i}{2\pi}\right)^3\int_0^1 dt\ \mathrm{tr}(AF_t^2)\ , \tag{E.40}$$

$$\Omega^1_4(c, A) = \frac{i}{48\pi^3}\mathrm{tr}\left\{c\ d\left[AF - \frac{1}{2}A^3\right]\right\}\ . \tag{E.41}$$

Comparison with our example of the non-Abelian anomaly (E.32) shows that indeed

$$G[c; A] = -2\pi\int \Omega^1_4(c, A)\ . \tag{E.42}$$

[4] A more precise definition of the Chern character is the following. Let Let E be a complex vector bundle over M with gauge group G, gauge potential A and curvature F. Then $\mathrm{ch}(A) := \mathrm{tr}\ \exp\left(\frac{iF}{2\pi}\right)$ is called the total Chern character. The jth Chern character is $\mathrm{ch}_j(A) := \frac{1}{j!}\mathrm{tr}\left(\frac{iF}{2\pi}\right)^j$.

Having established the relation between certain polynomials and solutions to the Wess-Zumino consistency condition using the BRST operators it is actually convenient to rewrite the descent equations in terms of gauge transformations. Define

$$G[\epsilon; A] := \int d^4x \; \epsilon_a(x) G_a[x; A] \; . \tag{E.43}$$

From (E.29) it is easy to see that we can construct an anomaly from our polynomial by making use of the descent

$$\mathrm{ch}_{n+1}(A) = d\Omega_{2n+1}(A) \quad , \quad \delta_\epsilon \Omega_{2n+1}(A) = d\Omega^1_{2n}(\epsilon, A), \tag{E.44}$$

where $\delta_\epsilon A = D\epsilon$. Clearly we find for our example

$$\Omega^1_4(\epsilon, A) = -\frac{i}{48\pi^3} \mathrm{tr} \left\{ \epsilon \; d \left[AF - \frac{1}{2} A^3 \right] \right\} \; . \tag{E.45}$$

and we have

$$G[\epsilon, A] = 2\pi \int \Omega^1_4(\epsilon, A) \; . \tag{E.46}$$

We close this section with two comments.

• The Chern character vanishes in odd dimension and thus we cannot get an anomaly in these cases.

• The curvature and connections which have been used were completely arbitrary. In particular all the results hold for the curvature two-form R. Anomalies related to a breakdown of local Lorentz invariance or general covariance are called gravitational anomalies. Gravitational anomalies are only present in $4m + 2$ dimensions.

E.2 Anomalies and index theory

Calculating an anomaly from perturbation theory is rather cumbersome. However, it turns out that the anomaly $G[x; A]$ is related to the index of an operator. The index in turn can be calculated from topological invariants of a given quantum field theory using powerful mathematical theorems, the Atiyah-Singer index theorem and the Atiyah-Patodi-Singer index theorem[5]. This allows us to calculate the anomaly from the topological data of a quantum field theory, without making use of explicit perturbation theory calculations. We conclude, that an anomaly depends only on the field under consideration and the dimension and topology of space, which is a highly non-trivial result.

Indeed, for the operator $i\gamma^\mu D_\mu$ appearing in the context of the chiral anomaly the Atiyah-Singer index theorem (c.f. appendix B.4 and theorem B.45) reads

$$\mathrm{ind}(i\gamma^\mu D_\mu) = \int_M [\mathrm{ch}(F) \hat{A}(M)]_{\mathrm{vol}} \; . \tag{E.47}$$

[5] The latter holds for manifolds with boundaries and we will not consider it here.

We studied the chiral anomaly on flat Minkowski space, so $\hat{A}(M) = \mathbb{1}$. Using (B.51) we find

$$\mathrm{ind}\,(i\gamma^\mu D_\mu) = -\frac{1}{8\pi^2}\int \mathrm{tr}F^2 \ . \tag{E.48}$$

and

$$G[x;A] = \frac{1}{16\pi^2}\mathrm{tr}[\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}(x)F_{\rho\sigma}(x)] \ , \tag{E.49}$$

which is the same result as (E.17). We see that it is possible to determine the structure of $G[x;A]$ using the index theorem.

Unfortunately, in the case of the non-Abelian or gravitational anomaly the calculation is not so simple. The anomaly can be calculated from the index of an operator in these cases as well. However, the operator no longer acts on a $2n$-dimensional space, but on a space with $2n+2$ dimensions, where $2n$ is the dimension of space-time of the quantum field theory. Hence, non-Abelian and gravitational anomalies in $2n$ dimensions can be calculated from index theorems in $2n+2$ dimensions. Since we do not need the elaborate calculations, we only present the results. They were derived in [13] and [12] and they are reviewed in [11].

We saw already that it is possible to construct solutions of the Wess-Zumino condition, i.e. to find the structure of the anomaly of a quantum field theory, using the descent formalism. Via descent equations the anomaly $G[c;A]$ in dimension $2n$ is related to a unique $2n+2$-form, known as the *anomaly polynomial*. It is this $2n+2$-form which contains all the important information of the anomaly and which can be calculated from index theory. Furthermore, the $2n+2$-form is unique, but the anomaly itself is not. This can be seen from the fact that if the anomaly $G[c;A]$ is related to a $2n+2$-form I, then $G[c,A] + sF[A]$, with a $2n$-form $F[A]$ of ghost number zero, is related to the same anomaly polynomial I. Thus, it is very convenient, to work with anomaly polynomials instead of anomalies.

The only fields which can lead to anomalies are spin-$\frac{1}{2}$ fermions, spin-$\frac{3}{2}$ fermions and also forms with (anti-)self-dual field strength. Their anomalies were first calculated in [13] and were related to index theorems in [12]. The result is expressed most easily in terms of the non-invariance of the Euclidean quantum effective action X. The master formula for all these anomalies reads

$$\delta X = i\int I^1_{2n} \ , \tag{E.50}$$

where $dI^1_{2n} = \delta I_{2n+1} \quad , \quad dI_{2n+1} = I_{2n+2}$. The $2n+2$-forms for the three possible anomalies are

$$I^{(1/2)}_{2n+2} = -2\pi\left[\hat{A}(M_{2n})\ \mathrm{ch}(F)\right]_{2n+2} \ , \tag{E.51}$$

$$I^{(3/2)}_{2n+2} = -2\pi\left[\hat{A}(M_{2n})\ \left(\mathrm{tr}\ \exp\left(\frac{i}{2\pi}R\right) - 1\right)\ \mathrm{ch}(F)\right]_{2n+2} \ , \tag{E.52}$$

$$I^{A}_{2n+2} = -2\pi\left[\left(-\frac{1}{2}\right)\frac{1}{4}\ L(M_{2n})\right]_{2n+2} \ . \tag{E.53}$$

To be precise these are the anomalies of spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ particles of positive chirality and a self-dual form in Euclidean space under the gauge transformation $\delta A = D\epsilon$ and the local Lorentz transformations $\delta\omega = D\epsilon$. All the objects which appear in these formulae are explained in appendix B.4.

Let us see whether these general formula really give the correct result for the non-Abelian anomaly. From (E.8) we have $\delta X = -i\int \epsilon(x)G[x;A] = -iG[\epsilon;A]$. Next we can use (E.46) to find $\delta X = -2\pi i\Omega_4^1(\epsilon, A)$. But $-2\pi\Omega_4^1(\epsilon, A)$ is related to $-2\pi \mathrm{ch}_{n+1}(A) = -2\pi[\mathrm{ch}(F)]_{2n+2}$ via the descent (E.44). Finally $-2\pi[\mathrm{ch}(F)]_{2n+2}$ is exactly (9.31) as we are working in flat space where $\hat{A}(M)$=1.

The spin-$\frac{1}{2}$ anomaly[6] is often written as a sum

$$I^{(1/2)} = I^{(1/2)}_{gauge} + I^{(1/2)}_{mixed} + nI^{(1/2)}_{grav}, \tag{E.54}$$

with the pure gauge anomaly

$$I^{(1/2)}_{gauge} := [\mathrm{ch}(A)]_{2n+2} = \mathrm{ch}_{n+1}(A)\ , \tag{E.55}$$

a gravitational anomaly

$$I^{(1/2)}_{grav} = [\hat{A}(M)]_{2n+2}\ , \tag{E.56}$$

and finally all the mixed terms

$$I^{(1/2)}_{mixed} := I^{(1/2)} - I^{(1/2)}_{gauge} - nI^{(1/2)}_{grav}\ . \tag{E.57}$$

n is the dimension of the representation of the gauge group under which F transforms.

Anomalies in four dimensions

There are no purely gravitational anomalies in four dimensions. The only particles which might lead to an anomaly are chiral spin-1/2 fermions. The anomaly polynomials are six-forms and they read for a positive chirality spinor in Euclidean space[7]

$$I^{(1/2)}_{gauge}(F) = -2\pi\ \mathrm{ch}_3(A) = \frac{i}{(2\pi)^2 3!}\mathrm{tr}F^3\ . \tag{E.58}$$

The mixed anomaly polynomial of such a spinor is only present for Abelian gauge fields as $\mathrm{tr}(T_a)F_a$ vanishes for all simple Lie algebras. It reads

$$I^{(1/2)}_{mixed}(R, F) = -\frac{i}{(2\pi)^2 3!}\frac{1}{8}\mathrm{tr}R^2\mathrm{tr}F = \frac{1}{(2\pi)^2 3!}\frac{1}{8}\mathrm{tr}R^2F^a q_a\ . \tag{E.59}$$

[6] We use the term "anomaly" for both $G[x;A]$ and the corresponding polynomial I.

[7] Note that the polynomials are real, since we have, as usual, $A = A_aT_a$ and T_a is anti-Hermitian.

Anomalies in ten dimensions

In ten dimensions there are three kinds of fields which might lead to an anomaly. These are chiral spin-3/2 fermions, chiral spin-1/2 fermions and self-dual or anti-self-dual five-forms. The twelve-forms for gauge and gravitational anomalies are calculated using the general formulae (9.31) - (9.33), together with the explicit expressions for $\hat{A}(M)$ and $L(M)$ given in appendix B.4. One obtains

$$\begin{aligned}
I^{(1/2)}_{gauge}(F) &= \frac{1}{(2\pi)^5 6!}\mathrm{Tr}F^6 \\
I^{(1/2)}_{mixed}(R,F) &= \frac{1}{(2\pi)^5 6!}\left(\frac{1}{16}\mathrm{tr}R^4\mathrm{Tr}F^2 + \frac{5}{64}(\mathrm{tr}R^2)^2\mathrm{Tr}F^2 - \frac{5}{8}\mathrm{tr}R^2\mathrm{Tr}F^4\right) \\
I^{(1/2)}_{grav}(R) &= \frac{1}{(2\pi)^5 6!}\left(-\frac{1}{504}\mathrm{tr}R^6 - \frac{1}{384}\mathrm{tr}R^4\mathrm{tr}R^2 - \frac{5}{4608}(\mathrm{tr}R^2)^3\right) \\
I^{(3/2)}_{grav}(R) &= \frac{1}{(2\pi)^5 6!}\left(\frac{55}{56}\mathrm{tr}R^6 - \frac{75}{128}\mathrm{tr}R^4\mathrm{tr}R^2 + \frac{35}{512}(\mathrm{tr}R^2)^3\right) \\
I^{(5-form)}_{grav}(R) &= \frac{1}{(2\pi)^5 6!}\left(-\frac{496}{504}\mathrm{tr}R^6 + \frac{7}{12}\mathrm{tr}R^4\mathrm{tr}R^2 - \frac{5}{72}(\mathrm{tr}R^2)^3\right).
\end{aligned} \tag{E.60}$$

The Riemann tensor R is regarded as an $SO(9,1)$ valued two-form, the trace tr is over $SO(1,9)$ indices. It is important that these formulae are additive for each particular particle type. For Majorana-Weyl spinors an extra factor of $\frac{1}{2}$ must be included, negative chirality spinors (in the Euclidean) carry an extra minus sign.

Part IV

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